Beyond the Cognitive: The Other Face of Mathematics

1. Introduction

When I talk here of mathematics, I refer to "mathematical activity" or, to use the felicitous phrase of Davis and Hersh, "the mathematical experience." Since I am addressing an audience specifically interested in mathematics education, I shall refer mainly to the mathematical experience of children. However, most of what I have to say applies equally to the professional mathematician. I share with Piaget a belief in the heuristic value of trying as hard as one can to understand as much as one can of children's mathematics and mathematicians' mathematics in the same categories. Doing so can illuminate both sides.

The mathematical experience is a complex thing which must be understood from many points of view. I shall distinguish mathematical points of view from psychological points of view—and I shall make some further distinctions within the psychological. The first cut corresponds to a line which I think is ultimately untenable though heuristically valuable: a cut that separates the study of knowledge from the study of feelings, a cut between "the cognitive" and "the affective." A second cut distinguishes two ways to think about feelings. On one side are considerations like "that's fun" or "I hate math" which, as a first approximation, one might say have to do with what the feelings feel like. The other has to do with a more "structural" view of feelings. "Mathematics is a sublimation of libidinal conflict" would belong here if we wanted to say such things (which I don't).

More to the point of my remarks is a growing conviction that the psychology of mathematical thinking stands to gain by using ideas from other psychological contexts—in particular, that our understanding of the relationship between people and mathematical objects can be clarified by concepts used to understand internalized human objects. Indeed, a research perspective emerging from my collaboration with Sherry Turkle suggests that the study of how people identify with mathematical objects may be the key to the outstanding problems in mathematics education. At the very least, it adds a new dimension to the classification of styles of mathematical thinking. Thinking in images vs. thinking in words is a well-established attempt to describe a dichotomy of styles. Turkle and I suggest that modes of identification with internalized objects might be even more important. "Do you observe the mathematical scene in your head or are you in it?" is just one example of the kinds of questions that come up here. Pursuing this theme suggests some new sources of ideas for the study of the psychology of mathematics education—among these a new wave of feminist studies such as the work of Evelyn Fox Keller and of Carol Gilligan.

2. Examples

The central points I want to make are better grasped through examples than through generalizations. I shall draw most of the examples from a common source—the Logo turtle. But this is not a lecture about turtles. The turtle's role here is to carry some much more general ideas.

I shall develop the principal idea I want to present at the PME meeting as the fourth of four views of work with turtles. I have found it a valuable experience to separate these views and then to bring them back together. I am going to ask you to follow my experience by concentrating on each in turn.

A mathematical view of the turtle

Euclid built his geometry on the notion of a point, an entity whose only property is its position. (Euclid explicitly says it has position but no magnitude, but we understand that it also has no color, no smell, no nothing except position.) A turtle is almost but not quite as stripped down as Euclid's point; it has two properties: a position and a heading.

We allow our turtle to change its position, leaving behind it a trail of points, and to change its heading. However, the turtle and its trail cannot simply "jump around" in any direction. Its position may change only in such a way that the trail goes in the direction of the heading.

Such thinking allows us to build a mathematical theory in the same sense that Euclid's geometry or Newton's calculus is a mathematical theory. In this theory, we make constructions such as the angle between two curves and the total turn of a turtle trip. (Intuitively the same as the line integral of the turtle's changes in direction.) Using these constructions, we prove theorems such as the Total Turtle Trip Theorem—which says that the total turn of a closed curve is a multiple of 360 (measured in degrees) and exactly 360 if the path does not cross itself.

Many Euclidean propositions can be seen in a different light as special cases of turtle theorems. The first and best known example is the proposition about the sum of the angles of a triangle. The Total Turtle Trip Theorem says that the sum of what a Euclidean geometer would call the external angles of *any* polygon is 360. (And, for that matter, of any curved figure as well.) This is not quite the same proposition as Euclid's but serves the same purposes in giving one a way to think about the angles of a triangle. And in itself, I think it is a "better" theorem in ways that are characteristic of what one can do in turtle geometry: it is more general, more perspicuous, and more powerful all at once.

To get you into the spirit of it, I shall give you an example of a related case where the Euclidean proposition can be proven in a simpler way and a more general form with turtle geometry. In Euclid, the angle between a chord of a circular arc and the tangent at one of its ends is half the angle subtended by the chord at the center. The corresponding turtle theorem says that the angle between the chord and the arc (we allow angles between curves!) is half the total turn of the arc. (It is easy to see that total turn of a circular arc is the same as the angle at the center of the circle.)

For me, a first layer of pleasure in this new view of the theorem comes from its simpler statement and less cluttered diagram. (You only have to draw the arc and the chord. The rest of the circle—lines drawn to the center and the tangent—are all unnecessary to the statement and actually hinder the proof.) Extra pleasure comes from the fact that it has a proof that is immediately obvious to anyone used to making turtle trips. And still more pleasure comes from generalizing to the following proposition.

Draw a line (corresponding to the chord). Draw a wavy (not circular) curve from one end of the line to the other. The total turn of the curve is equal to the sum of the two angles the curve makes with the line. The proposition about circular arcs is the special case where the turning of the turtle is distributed uniformly along the curve. Because uniform implies symmetrical we need only refer to one of the (two equal) angles in the circular case, but we see now that the theorem is "really about" the sum of the two angles (which are, in general, not equal).

Another layer of pleasure comes from recognizing an "opposite extreme case." The opposite of "distributed uniformly" is "concentrated at one point" which would mean that the "curve" is made up of two straight segments (call them the "sides") which form a triangle with the chord (call it the "base"). The total turn of this curve is the external angle between the two sides. Thus the proposition asserts that this external angle is equal to the sum of the two (opposite) internal angles at the base. (Familiar?) So two Euclidean theorems I have known all my life turn out to have a close relationship.

This kind of play is pure mathematics—working the mathematical theory from the inside. If you like it and want to do some more yourselves, you might try to anticipate how I'll use the proposition about chords and curves to give in my lecture a one-line proof of the Euclidean theorem about the angles subtended by a chord at the circumference of a circle.

A cognitivist view of the turtle

The pro-turtle remarks in the preceding section reflect mathematical values—my mathematical taste. If you disagreed with me, I might try to win you over by doing mathematics with you. Maybe I'd invite you to work through some problems with me or to look at some more propositions and proofs in turtle geometry. I might draw you into discussion about analogous situations in the history of mathematics; for example, I might draw your attention to my opinion that turtle geometry captures some of the essential ideas of classical differential geometry. All these are examples of the kinds of suasions that are normal in the culture of mathematicians.

Now let me change hats. I shall stop talking as a mathematician and start talking as a cognitive psychologist. From this point of view, a different kind of argument must be used to support advocacy of the turtle: for example, theories about knowledge and how it is represented. The biggest change of all is that we now have to refer to experiments, rather than to logic, in order to see whether our theories work. The following propositions are typical of the kind of "cognitivist" statement I am inclined to make and for which I have been collecting evidence over the years.

In the mathematical context, the turtle's merits had to do with what one could do with it mathematically. These merits are still relevant but new ones come into play, in particular the fact that heading and position as characteristics allow a person to identify with it, to anthropomorphize and "play turtle." Children know a lot about how to move around and can access this knowledge fairly easily by making or imagining body movements. By putting themselves in the place of the turtle, they can transfer some of this knowledge into the context of mathematical problems.

For example, an effective way for children to discover the facts of life in the world of triangles is to pretend to be a turtle: to walk around the triangle and be aware of their body movements. Thus, "play turtle" becomes a piece of heuristic advice. Moreover, it is not only good advice for learning to do geometry, it also allows a good prototype for the idea of heuristic, so it is a learning path into metacognition. Similarly, the Total Turtle Trip Theorem provides a particularly good prototype for the idea of theorem. It is a learning path into the culture of formal mathematics.

A first view of feelings

We move into yet another kind of discourse when we look at children's feelings about their work with the turtle. We are still in the domain of empirical psychology. Indeed, here we are in the clearest zone for experimental observation, for we can see how much children like doing this work. They tell us, if we ask them, that working with turtles is "more fun" than doing mathematics in their class workbooks. And they confirm their assertion by putting in long hours in which they seem to get pleasure from putting pictures on the screen, while their teachers get pleasure from the thought that in doing so, they are exercising mathematical and other desirable skills.

I shall pass quickly over this aspect of the psychology of feelings (including negative ones such as those commonly associated with the terms "math anxiety" and "math phobia") in order to move onto the next view which is the central one for this lecture. Indeed, I mention the psychology of feelings in the sense of what people like and don't like mainly as a foil to present something different.

A deeper side of feelings

Jonas Salk, best known as the originator of the polio vaccine, sees as a key to his success as a virologist the use of a strategy of pretending to be the virus he was trying to understand. Evelyn Keller in her study of Barbara McClintock talks at length about how McClintock would (subjectively) enter the cells she was studying.

I have mentioned three bases for my advocacy of the turtle: the first referred to mathematical values, the second to modes of gaining access to knowledge, and the third to the fun of making turtle drawings. Here I focus on a fourth—the psychology of the kind of process that Salk and McClintock talk about.

These phenomena are not easy to study. When you watch children at work with the turtle you can sometimes pick up in their body movements signs that they are mentally going through the motions of the turtle. But more structured information has to come from other sources.

For example, at the Hennigan School in Boston, besides collecting much more data about children's behavior with turtles, we are looking at other areas of work for children with promise as windows onto this kind of issue. For example, in our Lego/Logo project, we study how children work with—both construct and think about—mechanisms. An interesting case is understanding gears. If you look at them from the outside, you are exposed to the temptation of thinking in terms of general propositions such as "big gears are slow and strong." If you "get inside" the gear system, you come more easily to thinking of the actual interactions of the teeth. Since we suspect that there is a bias (only a bias, not a law) for girls to be more inclined to identify with objects of thought, we have a slightly counter-cultural situation in which girls have an advantage in understanding a mechanism.

Of course it is always good to ask children their opinions and they will sometimes tell a skillful interviewer much about how they relate to the turtle as they work. Some place themselves in the turtle's world. Some stay out and view the turtle "objectively" as an external object on the screen. Some of those who enter the turtle world say that they "are" the turtle while others talk as if they are "with" the turtle but separate from it and telling it what to do. In *The Second Self*, Turkle describes a boy who likes to place himself in constructs, such as space ships, made out of the multiple turtles available in the Logo system he was using. He would be the pilot of the space ship. The boundary between himself and the turtles was firm.

Boundaries between self and non-self is an area that has been studied over many years. Can we draw on these invesigations to deepen our understanding of mathematics education?

An example that is beginning to yield rich results in our work at Hennigan attempts to follow the lead that extremely obsessional people tend to develop defenses through separateness of the self and through highly controlled manipulations of objects. Expecting them to apply the "play turtle" idea in a highly identifying way would be like expecting "an old dog to learn new tricks." So one might expect to find a correlation between behavior like the spaceship pilot mentioned above and obsessional personality traits. Preliminary results seem to be confirming (and enriching) this expectation.

On a deeper—and by the same token more speculative—level, we are looking at ideas about the object identification and self boundaries in the context of early development of personality. Keller, Gilligan, and Turkle all pick up in slightly different forms an idea from the object-relations branch of psychoanalytic theory: other things being equal (which they seldom are, of course), boys would be expected to make a sharper separation of self from other since they can use their gender to symbolize their separation from their mother. This sharper separation, as well as its involvement with a sense of gender, could be expected to mark intellectual style throughout life. Does it affect mathematical learning? Does this suggest that children with different patterns of identification should be allowed to approach mathematics through possibly very different learning paths?

3. Conclusions

Piaget saw affect as a kind of motor that propelled but did not shape intellectual development. Most motivational theorists adopt a similar perspective: the nature of mathematics is given, the role of motivational theory is to understand the conditions under which children will like it enough to learn it. The "math anxiety" therapeutic movement sees failure to love and learn mathematics as a kind of neurosis to be "cured" by a kind of therapy.

But let's suppose (just to be fanciful) that some patterns of personality development make it natural to learn in a turtle-like fashion, and other patterns make a Euclid-like geometry more natural. Would this not suggest supporting the natural path of development by allowing alternative mathematical cultures in which the content of the mathematics they learned was different? And in the extreme case, radically different? I asked above whether children should approach mathematics by very different paths. But this form of the question supposes a common end. My provocative conclusion will be to sketch a world in which we recognized the right to difference in the end as well as in the means. The therapeutic approach of the math-anxiety specialists would be reversed: it is not neurotic but rather a highly coherent act of self-expression to resist learning in a way that goes against the grain of the deepest structures in one's own self.