The Mandelbrot Set Fractal as a Benchmark for Software Performance and … Human Creativity

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Abstract

Science and art are considered as distinct areas of the global gamut of human activities. Scientists draw inspiration from art and artists embed science in their work. This paper presents the results of a one-year experiment, which started from testing a dialect of Logo, resolved some mathematical and programming challenges of dynamical systems with complex numbers and ended up as an artistic computer-graphics exhibition in Sofia University.

The paper begins with a short historical and mathematical description of the Mandelbrot set fractal, two small fragments of which is shown in Figure 1. A program generating the fractal image is provided as pseudo code as well as a short discussion about the colour schemes.

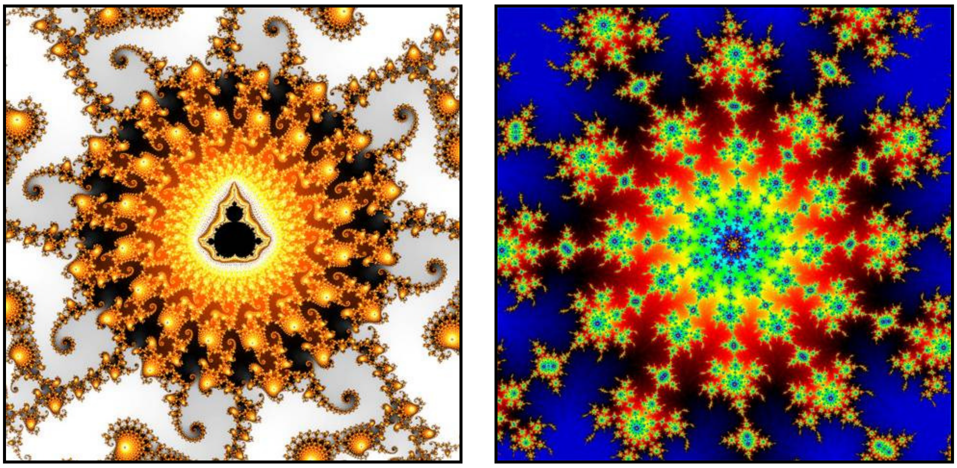

Figure 1. Two fragments from the Mandelbrot set fractal

The focus of the paper is on making of the gallery, on the ideas that have been considered and the decisions that have been made. Several interesting observations are also discussed. They include a multidisciplinary research and a collaborative work of a group of unknown people. Except for an exhibition, the result of this work is the acknowledgement that:

- People want to create and want to share their creations.
- It is challenging to try new ideas and approaches. To be creative, someone needs to have the anxiety to try something that he/she has never done before

This paper is an illustration how science and art can fit together in a mutually enriching way following the core ideas of learning by creating tangible artefacts and gaining new knowledge and skills.

Keywords

Fractal, Mandelbrot set, science and art

Introduction

Being the first international conference dedicated predominantly to Constructionism, it is likely that many papers will focus on Constructionism from the educator point of view. To become conscious of this learning theory and its practical applications, the author decided to submit himself to an experiment where he will play the role of a student learning through the Constructionism approach. This role would require **learning an entirely new set of multidisciplinary ideas**, **collaborating with unknown so far peers** from all over the world and **producing tangible artefacts** with application in scientific, artistic and educational contexts.

A fractal beginning

The Mandelbrot set fractal

The mathematician Benoît Mandelbrot was the first person who used a computer to visualize the behaviour of a dynamic system. In 1975 he introduced the word *fractal* (from the Latin *fractus*, *broken*) to denote objects with fractional dimension (Mandelbrot, 1983). His research built the foundation of fractal geometry – the link between classical Math and the "chaos" of atmospheric turbulence, biological populations and the stock market. Decades before fractals were named, other mathematicians had studied them – Weierstrass, Koch, Lévy, Cantor, Poincaré and Julia.

The fractal found by Mandelbrot is called *the Mandelbrot set* – Figure 2. This is a set of all points in the complex plane, for which the iterative application of a polynomial with an initial value of 0 generates a bounded sequence. In fact, a fractal is the boundary of the set. The colouring is determined by the speed at which the sequence crosses a preselected limit, beyond which there are no members of the Mandelbrot set (Mandelbrot, 2004). Mathematically, the Mandelbrot set M is defined as the set of all points with a finite supremum (i.e. least upper bound) of $f^n(0)$ where $n \to \infty$ and $f(z) = z^2 + c$.

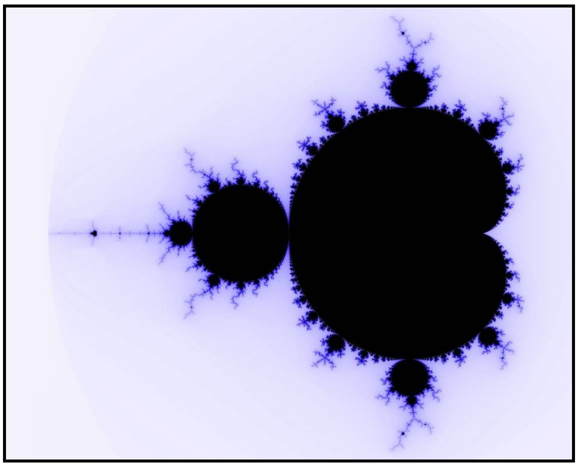

Figure 2. The Mandelbrot set fractal

Software implementation

The generation formula for the Mandelbrot set is not directly convertible into a Logo program code, because most Logo dialects do not support complex numbers. The solution is to use the geometrical representation of complex numbers – each of them corresponds to the coordinates of a point in the plane.

Another problem occurs with the implementation of the supremum. It involves infinity and it is practically impossible to do infinite number of calculations for the generation of a single point of the set. Fortunately, there is a mathematical proof that in many cases it is possible to predict whether the supremum is infinite. For example, if the distance between point z and point $(0,0)$ is 2 or greater, then z will go into infinity as we continue to do more iterations. For such point there is no sense to continue with more iterations – this point does not belong to the Mandelbrot set.

Sometimes, for a given c we need a few thousands of iterations, for another c we may need billions, and yet, there are values for which we can never cross the boundary set by threshold 2. This is the reason for the introduction of a limit of iterations. If we reach the limit and z is still within the boundary, then we assume that z will always stay inside (and thus, z will be a member of the Mandelbrot set).

The algorithm generating the Mandelbrot set can be expressed in pseudocode as:

```
1: For each pixel (x0,y0) do:
2:    n,x,y ← 0
3:    while n<max do:
4:       if x²+y²>2² then: plot(x0,y0,n) stop
5:       x´ ← x²-y²+x0
6:       y  ← 2xy+y0
7:       x  ← x´
8:       n  ← n+1
9:    plot(x0,y0,0)
```

Line 3 ensures that the program will do a finite (at most max) number of iterations, line 4 checks the threshold, and lines 5 to 7 implement the calculation $z \leftarrow z^2+c$. The procedure *plot* is used to draw a single pixel at coordinates (x_0,y_0). The third input of *plot* is the colour, which is usually set to the number of iterations n. If we select a colour gradient that extends from 0 to $max-1$, then some details might be invisible – they will appear as indistinguishable colours. For example, colours corresponding to n=350 and n=400 in Figure 3 will both look blue if the colour gradient spans over the whole interval from 0 to 1000. However, if the colour gradient is shorter (e.g. 200) and tiles until it reaches 1000, then the colours of n=350 and n=400 will be quite different. It is a matter of aesthetical judgment to decide the size of the colour span and the order of colours in it.

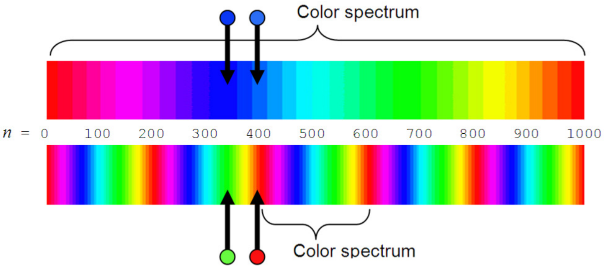

Figure 3. Colour gradient span interval of 1000 (up) and 200 (down)

This colour scheme is often referred to as *Escape Time Colouring*, because the colour depends on the speed of escaping from a circle of radius 2. The scheme produces visible colour bands for faster escapes. There are options to smooth the colours (Stevens, 2005) and one of them is the *Normalized Iteration Colouring*, where the colour is determined by $n + \log_2 \log_{f^n(0)} 4$.

First posters

In the autumn of 2008 it was decided to make several performance tests of the Lhogho[1]. Designed as a compiler, it was expected to outperform any other Logo in term of numeric processing. A benchmark should do millions of math operations and the calculation of the

[1] Lhogho web site: http://lhogho.sourceforge.net

Mandelbrot set loomed as a perfect candidate – it required an unavoidably huge amount of mathematical calculations for determining the colour of just a single pixel.

Attempts with other Logos needed more than 10 minutes to generate the simplest picture of the Mandelbrot set. Using Lhogho took just a few seconds without any code optimizations. When the benchmark program[2] was ready, it was decided to visualize the generated image in order to verify that all calculations were correct.

The first dozens of zoom-ins showed images, which strikingly resembled things in the material world, like a forest, a solar protuberance, and a coffee's cream. Some of the images were digitally manipulated and blended with real photographs. The results were sent to colleagues and their feedback was extremely eloquent – "*Make an exhibition!*"

The next six months were spent in finding new interesting fragments in the fractal and converting them into artistic pictures. Figure 4 is a map of all places selected for the exhibition.

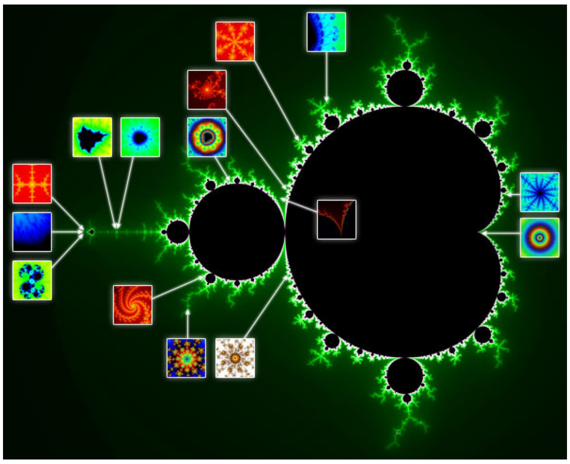

Figure 4. Points of interest

Observations

The actual making of the posters required answering many questions, like:

- What information must be included in each poster? How to present it?
- What real life object or phenomena can be illustrated?
- How to get a suitable photograph entirely legally and at a sufficient resolution?

Several interesting and somewhat unexpected observations were reached during the making of the exhibition. The next subsections describe some of them.

The depth of the fractal

All close-ups of the fractal showed that it has practically infinite levels of details. There are areas that can be zoomed in thousands and millions of times; and yet they continue to provide new and different shapes.

Figure 5 shows the initial fractal in the top left-most image. There is a small square in the middle. The content of this square is shown zoomed 10 times in the next image, then its centre is zoomed 10 times, and so on. The last image uses a magnification of 10,000,000 times. In some of the studies of the fractal, we had zoom factors of 1,000,000,000,000, which reached the precision limit of the mathematical libraries used for the calculations.

[2] Its source code is included in the Lhogho package and is licensed under GPL.

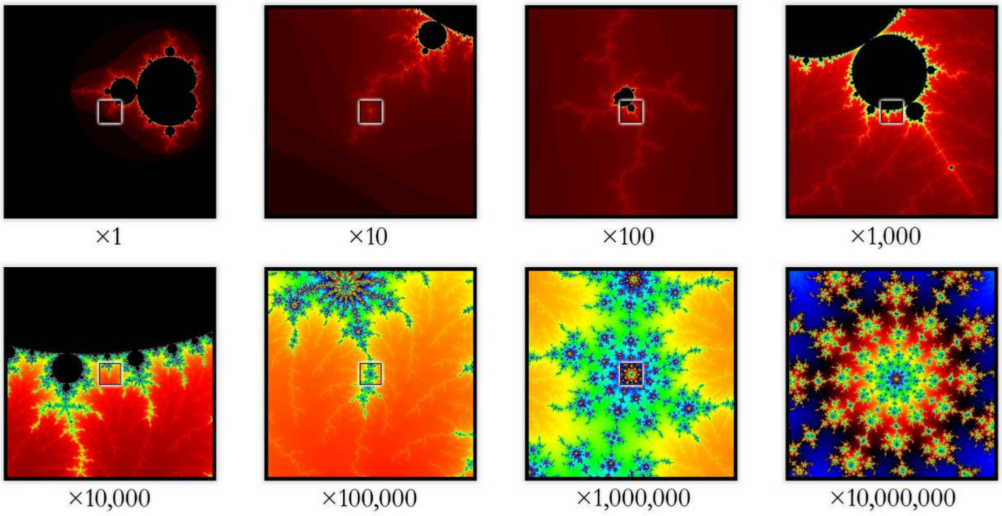

Figure 5. Zooming in up to 10 million times

One of the posters is dedicated to the unlimited diversity of the fractal – the middle poster in the 4th row in Figure 10. It says that if the zoomed fragment is real size, then the other end of the fractal will reach the Sun. A slightly higher calculation precision would provide zoom factors where the fractal will become larger than the known Universe.

Loading the posters with information

It was problem to decide what information to include in each poster. Two of the most important elements were the fractal image and its artistic interpretation. However, these two elements were not enough, because each poster should provide a unique viewpoint on something new and should present navigational data so that people can recreate the fractal if they want to. Figure 6 (left) shows what is "hidden" in each poster – there is the computer-generated fractal (1), the coordinates of its centre (2 and 3), its scale (4, 5 and 6). Thumbnails (7, 8 and 9) visualize the process of reaching the fractal image at steps by a scale factor of 10.

Image (11) is the artistic representation of the fractal. It shows a relation between mathematically defined (and computer generated) image with objects, events and ideas from our lives. Although sufficient by itself, each artistic representation is accompanied with a short text describing something interesting about the topic (10). Finally, a credit line (12) reveals the collaboration with other people.

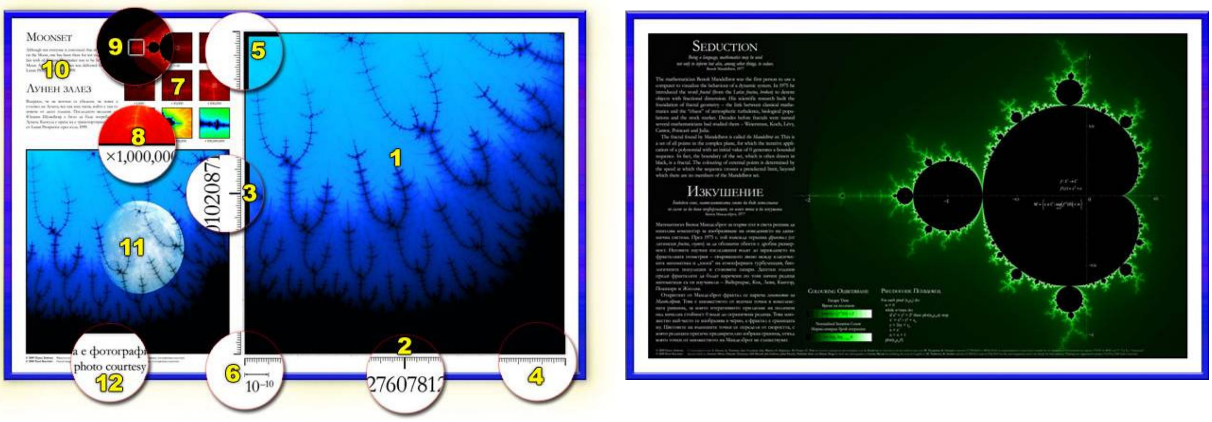

Figure 6. Information presented in each poster (left) and the introductory poster (right)

All posters use the same layout to make it easy for viewers to find what they are interested in – technical details, artistic interpretations or the additional interesting facts.

There is one additional introductory poster, Figure 6 (right), which shows a high-resolution image of the whole Mandelbrot set along with a brief history of the fractal and its generation methodology. This poster also contains the mathematical backbone of the fractal and a pseudo code of a program that draws it.

International, national and familial collaboration

The making of the posters required collaborative efforts of many people … at every step from the initial design to the final hanging of the posters in the exhibition hall. For example, people contributed with digital photographs, with ideas, with support for the physical making of the exhibition, with the translations and language tuning, and so on.

Some of the artistic fractal representations are based on digital photographs by other people. We identified the authors and they were from all over the globe – from Japan to Canada. They were all asked for permission to use their work. We did not know any of these people and the initial hopes were that approximately 10% of them would agree. The process of getting permissions was expected to be rather long. However, these hopes were groundless. Surprisingly, authors of the photographs gave permissions, so we say again *Thank you* to Annette Olson, Daisuke Tomiyasu, Elfi Berndl, Jon Sullivan, John French, Nicholas Gere and Simon Tong.

All posters are bilingual – the texts are both in English and Bulgarian. Louise Blyton from Australia and Svetla Boytcheva from Bulgaria provided valuable fine-tuning of the texts' contents and style.

A group of colleagues from the Sofia University and the Bulgarian Academy of Science (namely: M. Todorova, B. Sendov, D. Dobrev, E. Sendova, E. Stefanova, E. Kovatcheva, N. Nikolova) provided technical and moral assistance. Elitza Boytcheva, author's daughter, gave comments and ideas, as well as found factual bugs in the artistic images in some of the posters.

Figure 7. Prof. Dobrev and prof. Sendov assisting the exhibition installation

Learning in a multidisciplinary way

The preparation of each poster included the hunt for some interesting and little-known facts. It was quite challenging to browse hundreds of documents until a proper story was found. As a result, the posters now refer to a broad range of topics covering time, space and life.

- *Multidisciplinary topics*. The posters describe objects and events from various knowledge areas like Astronomy, Biology, Palaeontology, Physics, History, Geology, Mathematics, Meteorology, Geography, Engineering, Manufacturing, Jewellery, Crafts and Game design.
- *Time dimension*. The topics span from 100 million years in the past up to 5 billion years in the future of the Earth.

  

Figure 8. Hand-made elements

- *Life dimension*. There are posters about prehistoric life of spiders, wasps, bees, flies and mites, and yet there are posters about microorganisms and … extraterrestrial life.
- *History dimension*. Several of the posters present historical events and people – Eugene Shoemaker, who was buried in the Moon; the Chinese monk Li Tian, who made the first firecracker; Menaechmus, a tutor of Alexander the Great, who discovered the hyperbola and Appollonius, who named it.
- *Artefacts dimension.* This dimension shows some interesting facts about the objects around us. For example, a cup of coffee simulates turbulent atmospheric phenomena at a scale that is still not reachable by modern supercomputers; the zipper was initially used only for boots and children's clothing; and the left and right shoes did not exist two centuries ago, because of manufacturing problems.
- *Geographical dimension.* The facts in the posters refer to various locations on Earth, like Spain, where the oldest amber with three orders of flying insects was found; or the Tokara Islands in Japan, where the longest solar eclipse for this century was observed in 2009; or the night skyline of Hong-Kong.
- *Computer graphics dimension.* Most of the artistic interpretations of fractals are done by blending digital photographs with computer-generated images. This blending is a complex process, which requires various techniques. Some of the objects, however, are entirely artificial – see Figure 8. They have been model from scratch with an image manipulation program. For example, the jewellery shaped like the Mandelbrot set is entirely artificial; each piece of the 361-pieces puzzle is made from scratch; the pagoda is built by borrowing shapes from the fractal.

The exhibition

The initial ideas for an exhibition appeared in late 2008. Half a year later, the posters were almost complete (as image files). The last step was to print the posters, put frames and hang them somewhere. A suitable event was the 120th anniversary of the Faculty of Mathematics and Informatics at Sofia University.

The name of the exhibition was chosen to be "*Seduction*". Although it sounds too provocative, it is based on Benoît Mandelbrot's words:

> *"Being a language, mathematics may be used not only*
> *to inform but also, among other things, to seduce."*

With the help of colleagues and the financial support by the Faculty, the exhibition was opened in October 24, 2009. The first visitors were the people from the cleaning stuff who provided an initial feedback. An interesting but still unexplained observation was that many of the later visitors saw the exhibition in groups of four - Figure 9. A set of thumbnails representing all 15 posters (except the introductory one) is shown in Figure 10. The exhibition has an on-line version[3] for those, who cannot visit the real one. The online posters, however, do not demonstrate the full beauty of the fractals and the extreme level of details.

Figure 9. The exhibition and the visitors

Except for the permanent exhibition at the Faculty of Mathematics and Informatics, the posters are also included in a private collection in Australia. Individual posters are sent to science and educational institutions in North America. Mini-exhibitions have been set up for two nation-wide events – the *Conference of the Union of Bulgarian Mathematicians* and the *National IT Olympiad* for 5-12 grade students.

Lessons learned and future plans

One of the most important things that was learned for the last year is that the majority of the nowadays projects are results of collaborative efforts and utilize various resources from different places. People who create and construct scientific and artistic artefacts are happy to see that their work is being used by others, or at least that it inspires other people to be creative.

The other important lesson is not to be afraid of experimenting with ideas and techniques, which are entirely new to you. This exhibition is made by a person who has never made any other exhibition. Almost everything in the process was new – from requesting permissions for using photographs and working with a company for large-scale digital print, to putting frames on the posters and hanging them on the walls.

It would be perfect if the exhibition triggers the creation of various educational, scientific and artistic activities, which could be:

- Investigating the Mandelbrot set fractal and finding other interesting places
- Looking for relations between other fractal images and the real life
- Seeking additional information for interesting, but little known facts
- Practicing skills for digital image manipulation
- Developing new, faster algorithms for generating fractals
- And finally – exploring new directions for creativity

[3] On-line exhibition: http://mandelbrot-set.elica.net

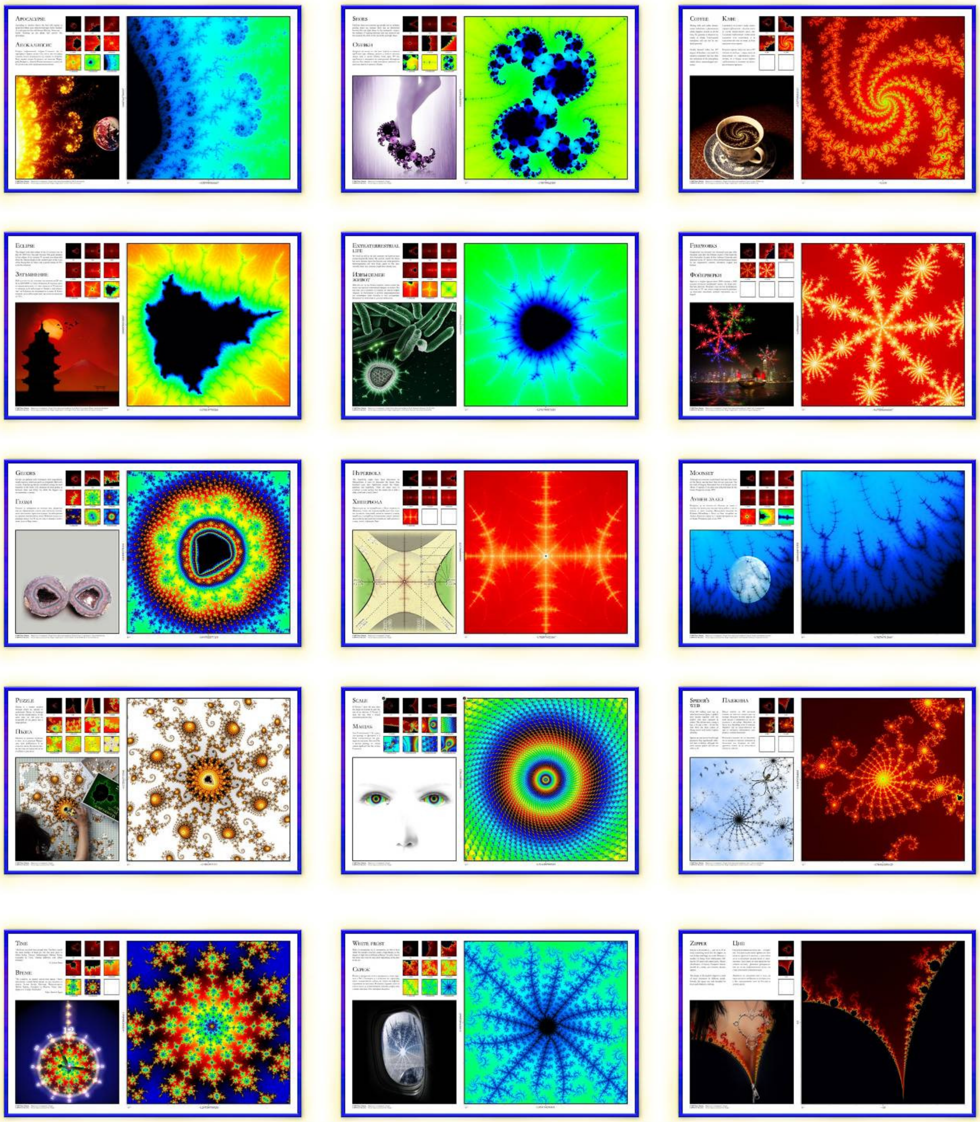

Figure 10. The full set of posters

Fortunately[4], it is hard to make any plans for the future. The exhibition was triggered by a boring work and it is impossible to imagine what element of the exhibition will trigger new ideas. That is perhaps the best lesson learned – even the most boring reality can turn on people's imagination and creativity. They only need to realize the precise moment and to capture the initial momentum. What happens next is something that cannot be planned and foreseen.

Postscript

It is author's belief that everything that is created as a result of some official or unofficial project is actually a (re)source of a new project. Similarly, the making of the exhibition is not a stand-alone effort and it initiateed a lot more activities. Science and art can mutually catalyze themselves and the bird view of the exhibition illustrates this – Figure 11.

[4] Yes, fortunately.

The development of Lhogho initiated the fractal exhibition, which is the main topic of this paper. The exhibition evolved into an award-winning animated film called "*A Journey in the Mandelbrot set*". It shows a virtual tour to the locations in the exhibition and it won the first place in the *Fractal Movies* section of the *Spring 2010 Fractal Art Contest* organized by FractalForums.com.

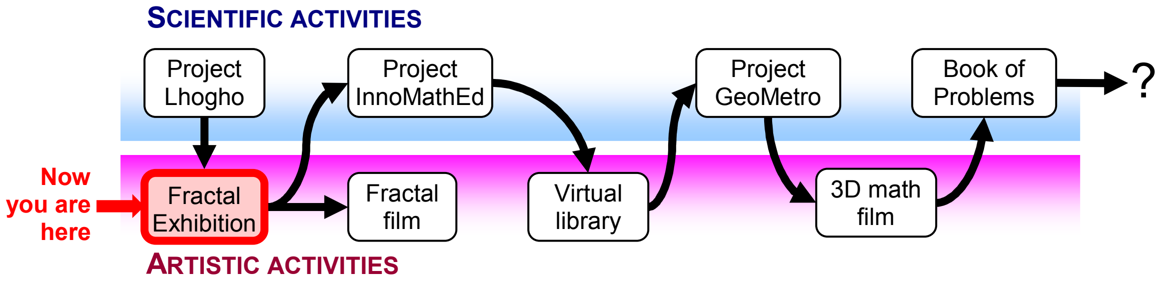

Figure 11. Mutual catalyzation of scientific and artistic activities in an educational context

One of the posters from the exhibition contains a mathematical problem about conical sections. This problem initiated the participation in the international project InnoMathEd[5] (Innovation in Mathematics Education on European Level) and a set of interactive 3D applications have been developed. This set gave birth to a virtual library with several dozens of 3D virtual models for drawing mathematical curves, modelling geometrical transformations, generating ruled surfaces and so on. Each model is accompanied with an animation, which is available to everyone[6].

The collection of animations inspired the beginning of a new project, called GeoMetro, which made the author to create another 3D film[7]. This math film shows seven different mechanisms for ellipse construction. Some of them are pictured in Figure 12. The plans for this film go beyond mere presentations. Its purpose would be to become the math problem conditions in a book of problems about ellipses. The conditions of the problems would be the film itself. This is a new and unique project which goal is to provide unprecedented motivation for students through 3D multimedia. The book of problems will be accompanied with a software library where mathematical virtual experiments could be done.

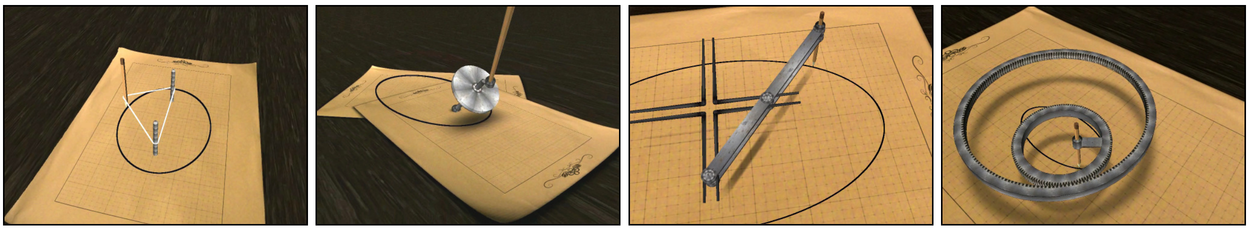

Figure 12. Snapshots from the 3D math film "Ellipses…"

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[5] Project InnoMathEd: http://www.math.uni-augsburg.de/prof/dida/innomath

[6] Mathematical devices: http://www.youtube.com/user/ElicaTeam#grid/user/6534E936D46257BF

[7] The 3D math film "Ellipses…": http://www.youtube.com/watch?v=1v5Aqo6PaFw