Early introduction to algebraic thinking in technological environments

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Abstract

A number of authors have investigated the feasibility of introducing young children to algebraic ideas: e.g. generalized arithmetic (Mason *et al.*, 1985); meaning of operations (Slavit, 1999); generalisation and progressive formalisation (Blanton and Kaput, 2002); algebra as a representation tool and resolution of problems (Da Rocha Falcặo, 1993); design of tasks to support algebraic thinking in elementary school (Blanton and Kaput, 2002); operating with the unknown (Carraher *et al.,* 2001); algebra from a symbolisation point of view (Kaput *et al.,* 2008) and the transition from arithmetic to algebra and the use of symbolic generalisations in a computer intensive environment (Tabach *et al.,* 2008).

This paper reports on an on-going study on early introduction to algebraic thinking in students of elementary school in Mexico, based on a teaching model that incorporates two routes of access: proportional reasoning and generalisation processes. The choice for the first route (proportional reasoning) is based on the familiarity that children have with this mathematical content at the fifth grade of elementary school. The second route examines the fact that the mathematical content is linked conceptually and historically (Radford, 1996) to functional variation. It's worth noting that at this level most of the students are in transit from additive to multiplicative thinking. The experimental work – which is currently in its second phase – involves paper-and-pencil, Logo and Expresser activities. For this, a teaching sequence was developed; pre- and post-questionnaires, as well as clinical individual interviews, were used to complement the data collection. For the construction of the teaching model, a theoretical framework was used that relies on the Local Theoretical Models (LTM) of Filloy (1990) and Filloy, Rojano and Puig (2008), and also takes into account Vygotsky's (1978) idea of Zone of Proximal Development (ZDP).

The study is situated at the end of the elementary school curriculum, in the fringe of pre-algebraic thinking, where algebraic syntaxes have not been introduced yet. In this study the algebraic ideas are introduced through a teaching sequence in two versions: pre-symbolic (perception of the idea of proportional variation) and symbolic in Logo and/or eXpresser environments (find and express a general rule, as well as incorporate it). The use of those technological environments, which have graphical, numerical and programming properties, can allow activities involving pattern recognition, where students can go from particular cases to expressing general rules and to testing those rules.

The first phase results have revealed that students are capable of understanding ideas of proportional variation, that they can discover a pattern and formulate a general rule, while they transit from additive to multiplicative thinking.

Keywords

Early algebra in primary school, generalisation processes, Logo

Introduction

Theoretical and empirical studies have shown that the transition from arithmetic to algebra is an important step needed to access more complex ideas in mathematics, and that a series of obstacles have to be overcome in order to master the notion of symbolic algebra. Some results suggest that it may be possible to overcome or avoid these obstacles depending on the way algebraic thinking is conceived and the way that early algebra is introduced in early stages. It is also believed that if the routes of access are familiar to students – such as proportional reasoning in primary school – and are specifically situated within the curriculum – for example within the 5th and 6th years of primary school – students are able to access early algebraic thinking even though their mathematical reasoning is in transit from additive to multiplicative reasoning. On the other hand, it is well known that didactical times for the learning of algebra are long and it seems appropriate to initiate students to early algebraic thinking at early ages (7-11 years old), taking advantage of the sources of meaning that the curriculum contents in primary school offer. As a reaction to these ideas, many authors have focused on research on early algebraic thinking through different perspectives, as cited in the abstract in the previous page.

A study for early introduction to algebraic thinking in technological environments

The study reported here is on early algebraic thinking but stands apart from the works previously cited, in that it proposes a conceptual route that does not break with numeric or algebraic thinking. It is an introduction to algebraic ideas where an algebraic symbolisation is not necessary reached, although sometimes it can be attained.

The research is situated at the end of the curriculum of primary school, in the layer of pre-algebraic thinking where the students have not yet been introduced to algebraic syntax. In the present study, algebraic thinking is approached from the point of view of proportional reasoning and generalisation processes. The main purpose is to develop an alternative route towards building a teaching model that allows students to transit from additive to algebraic thinking, incorporating sources of meaning such as proportional reasoning from the curriculum. Thus, algebraic ideas are introduced along two main lines:

1.- Pre-symbolic – using the idea of proportional variation and symbolic – where the general rule has to be found and expressed by means of a series of problems in a didactical sequence.

2.- Starting from proportional reasoning, which is considered a part of the multiplicative field, we develop further this mathematical idea towards proportional variation, variable as a functional relationship and general number by means of generalisation processes. Generalisation processes mean that students are involved in the detection of patterns, and it is helpful for them to represent the pattern by means of a rule, an entry point to the symbolic.

Going from particular cases to the general rules and testing them, can be achieved in practice by exercising pattern recognition in a graphical, numerical and programming environment where shapes-detection, similarity, repetition and recurrence can be easily manipulated, by the students themselves. We consider environments such as Logo and eXpresser as ideal for those purposes, as is discussed further down.

Aims of the study

Thus, the aims of our study can be summarised as follows:

- To study early algebraic thinking with students of the last years of primary school (grades 5 and 6) in technological environments of learning.

- To design and implement sequences of activities with digital technologies, exploring the two aforementioned routes of access to algebraic thinking: proportional reasoning and generalisation processes.
- To test several modalities of the use of digital technologies in the classroom.

The technological environments of learning

As stated above, for our project, two technological environments have been chosen: Logo and eXpresser. Many studies have investigated the potential of Logo for mathematical learning, including algebra learning (e.g. Hoyles and Sutherland, 1987, 1989; Ursini, 1993), including one of our own studies (Butto, 2005). In that latter study, the potential of Logo to facilitate the understanding of, specifically, proportional reasoning in 11-14 year-old children working collaboratively in pairs, was investigated.

We also like the fact that Logo gives children more autonomy on their own learning (Hoyles and Sutherland, 1989). Schoenfeld (1985; cited in Hoyles and Sutherland, 1989), stresses the role of meta-cognition when students are led to think on their own actions and thoughts; they assume self-control on their activities, are capable of taking their own decisions, can change their strategies and the way they organise and solve the problems. Logo is an ambient where heuristics and mathematical ideas are recreated (Noss, 1986). Thus, Logo creates a bridge between students' actions and their understanding of the mathematical relations that they require to write a program. In this way, children are capable of capturing their understanding in symbolic form and clarifying it with the aid of the computer.

In the present study, Logo was used in parts of the preliminary phase, which we present here.

For the second phase, we are in the process of developing activities for studying generalisations processes with eXpresser. The eXpresser microworld is a free java-based software of the MiGen project (*Intelligent Support for Mathematical Generalisation*; http://migenproject.wordpress.com/), which is led by Richard Noss and Alex Poulovassilis in the UK. This software "seeks to provide students with a model for generalisation that could be used as a precursor to introducing algebra" (Geraniou *et al.*, 2009). Following Papert's constructionist paradigm, eXpresser provides several approaches that allow students to construct their own mathematical models: in eXpresser, students can build figural patterns of square coloured tiles and express the rules underlying them. Thus, in this microworld, children can work in a numeric, geometric and programming environment, and make use of patterns or regularities correspondingly to their inputs when they create a program and make sense of what they are doing when they validate their predictions.

The design of the research and teaching sequence

The work presented here uses as theoretical framework the Local Theoretical Models (LTM) perspective proposed by Filloy (1990) and Filloy, Rojano and Puig (2008), which include four components: 1) the Teaching Model; 2) the Cognitive Processes; 3) the Formal Competence Model and 4) the Communication Model. In this project the focus is mainly on the first two: the teaching model and the cognitive process.

It is the aim to design a teaching model where, in the learning sessions, children work in several environments: paper-and-pencil, Logo (and later also with eXpresser), in order to cover the two alternative routes of access to algebraic thinking that we have proposed – as outlined before, and shown in Figure 1 – thru a teaching sequence that is applied as a mean to promote access to initial algebraic notions.

That is, we also use the Vygotskian idea of Zone of Proximal Development (ZPD) – which Vygotsky (1978) defines as the distance between the level of current child's development and the higher level of potential development – in that the didactic sequence is intended to help students in their development through their ZPD. It is thus important to determine the level of potential development and the level of current development. For that we explore and analyse the children's zone of current development and the evolution toward the first algebraic ideas through the application of pre- and post- questionnaire and ad-hoc interviews. These questionnaires and interviews give us insights into children's initial and later notions, and ZPDs, about proportional reasoning and generalisation processes.

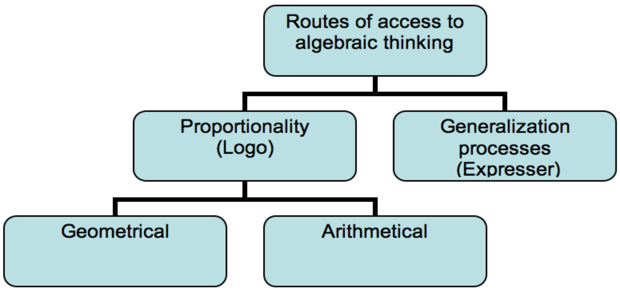


Figure 1: The two routes of access to algebraic thinking

In terms of the routes of access to algebraic thinking, we base the first route (proportional reasoning) on the familiarity that children have with this mathematical content at the fifth grade of elementary school. Another reason is the fact that the mathematical content is linked conceptually and historically to the idea of functional variation (Radford 1996). It is worth reminding the readers that at this level most of the students are in transit from additive to multiplicative thinking.

For the task design, we used Mason's (1985) idea about generalizing in algebra through four stages: 1) perceive a pattern, 2) express the pattern, 3) record the pattern and 4) test the validity of the formulas. Several researchers, such as Ursini (1993), assert that children find it difficult, when working with numerical patterns, to describe and express a pattern algebraically. Pegg (1990; cited in Durán Ponce, 1999) mentions that the discovery of patterns requires three processes: to experiment with numerical patterns; to express the rules by means of explanations and to encourage students to express their rules in a abridged way. Hoyles and Sutherland (1989) argued that the numerical and geometric environment of Logo allows children to observe numerical and geometric patterns and build general rules in algebraic or pre-algebraic terms.

Methodology

The study is being carried out with 20 students (10-11 years old) of the 5^{th} and 6th grades of elementary school. Students in this age-group tend to privilege mathematics contents belonging to the field of additive structures.

As stated above, (pre- and post-) questionnaires were designed to explore children's numerical thinking; specifically to explore proportional thinking and generalisation processes. The didactic sequences are meant to develop those algebraic thinking processes. The activities of the teaching sequence are carried out with paper-and-pencil, as well as some with Logo. (Note: In the phase reported here, we only use Logo; in a later phase, we will also incorporate eXpresser activities).

The contents of the questionnaires and activity sequences consist of problems related to proportional reasoning and/or generalisation, with variables as general numbers or in functional relationships. In many problems or activities, children are asked to complete tables of values. One questionnaire is on proportional reasoning, and involves problems such as identifying proportional figures; completing visual sequences that follow proportional patterns; constructing proportional figures; and problems with liquid proportions. Another questionnaire is on processes of generalisation and functional variation: the problems involve things like completing terms of arithmetic and geometric sequences; ordering "cards" of values of weights and heights of children; analysing the increase of production of a plastics machine (see Questionnaire Problem 4, below); monetary distribution amongst several people in different proportions; and completing sequences of figures.

We have two didactic activity sequences. The first is on proportional reasoning and includes activities that involve drawing, with Logo, different sizes of squares and of other figures (such as chairs and tables) keeping the proportions (and observing the similarities in the procedures); identifying figures that are in the same proportions (e.g. figures of persons, of tables); finishing a drawing of a car that is proportional to a given one.

The second didactic sequence is on generalisation processes and involves: * Drawing with Logo different sized letters (such as 'E's), first in a sequence, then writing a general procedure for any given size. * Completing sequences of figures and of polygonal numbers and finding the general rules. * A problem involving a horse race, where each horse starts at different time and runs at different speeds. * Drawing Logo squares in different sizes, observing the invariants and the variable values and writing a general procedure. * Experimenting with a recursive Logo procedure for drawing a tree with as many branches as given by a variable. * Experimenting with a Logo procedure that draws a spiral star, and uses 3 parameters.

In order to study social interaction during the working sessions of the didactic sequences, a model of mathematical discussion, consisting of the following four components, is used: 1) Individual and collective presentation of different solutions. 2) Individual reconstruction of the process of solution of a problem. Students mention their own strategies and abandon those that are not efficient. 3) Collective exposition of the new knowledge. Students are asked to share what they have learned, comparing situations and beliefs. 4) Institutionalisation of knowledge.

According to Tudge (1992), social interaction among pairs promotes information within the ZPD, promotes cognitive development and leads thinking in children to progress towards adult models within a cultural practice. In this process of collaboration, students learn meanings, behaviours and adult technologies.

After the working sessions, students are given a post-questionnaire, then the children are interviewed in order to verify the evolution of algebraic ideas and confirm the results obtained in the questionnaires and from their worksheets.

Answers to the initial questionnaire were categorised initially in levels of mathematical conception (high, medium and low):

- *High level*: is characterised by the comprehension of proportional reasoning, functional variation and generalisation processes. Thinking is in algebraic or pre-algebraic terms.
- *Medium level*: is characterised by a transitional thinking, which goes from the use of additive or multiplicative resolutions, either in proportional reasoning or generalisation processes. Transitional: from arithmetical to pre-algebraic thinking.
- *Low level*: is characterised by the use of purely additive strategies and students present difficulties in understanding problems of proportional reasoning as well as generalisation processes. Purely additive thinking.

The first level of analysis includes the strategies used in the solution of the problems (Logo and paper-and-pencil), obtained by the analysis of students' worksheets, and observations during the didactic sequence; these strategies were categorised as: arithmetic-multiplicative, incomplete-multiplicative and complete- multiplicative. The second level of analysis includes the social intervention in pairs during the working sessions and was classified according to Cobb (1994): univocal explanation, multivocal explanation, direct collaboration and indirect collaboration. The third level of analysis included the cognitive processes followed by the pairs in the solution of the problems by means of clinical interviews and cognitive maps with teaching. In general, the analysis of the longitudinal study is being done along two components of the LTM: the didactic and the conceptual models.

Results

Sample data from paper-and-pencil activities from the initial questionnaire

Questionnaire Problem 2

In the activity shown in Figure 2, children had to solve a word-problem dealing with mileage and gasoline consumption: in the first table of the column are the kilometres and in the second one the gasoline litres used. Below is part of the transcript (translated from the original Spanish) from the initial interview:

Interviewer: O.K. in question number 2 you had […]. How did you do it?

Child: With this (pointing to the first column), I thought that if 12 is half of 24, it had to be half the gasoline, and since 48 is the double of 24, it had to be the same below, the double of gasoline.

Interviewer: Why did you put in the relationship between the kilometres and the litres that it had to be "the 6th part"?

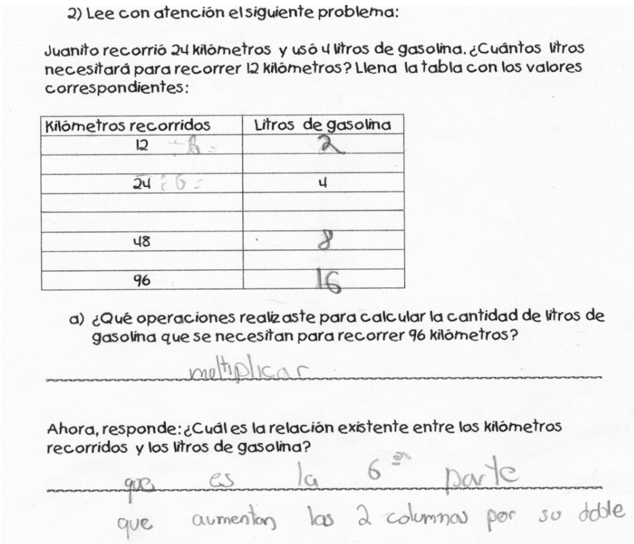
Child: Yes. Because I said: if 12 is divided by 2, its 6, then the 6th gives 2. If 24 is divided 6, it gives 4, and if 48 is divided by 6, it gives 8.

Interviewer: Why did you answer that it is the 6th part?

Child: Oh yes! I made a mistake…

Interviewer: How do they increase?

Child: Until here [points to a number] they increase by the double.



2) Lee con atención el siguiente problema:

Juanito recorrió 24 kilómetros y usó 4 litros de gasolina. ¿Cuántos litros necesitará para recorrer 12 kilómetros? Llena la tabla con los valores correspondientes:

Kilómetros recorridos	Litros de gasolina
12 ÷ 6	2
24 ÷ 6	4
48	8
96	16

a) ¿Qué operaciones realizaste para calcular la cantidad de litros de gasolina que se necesitan para recorrer 96 kilómetros?

multiplicar

Ahora, responde: ¿Cuál es la relación existente entre los kilómetros recorridos y los litros de gasolina?

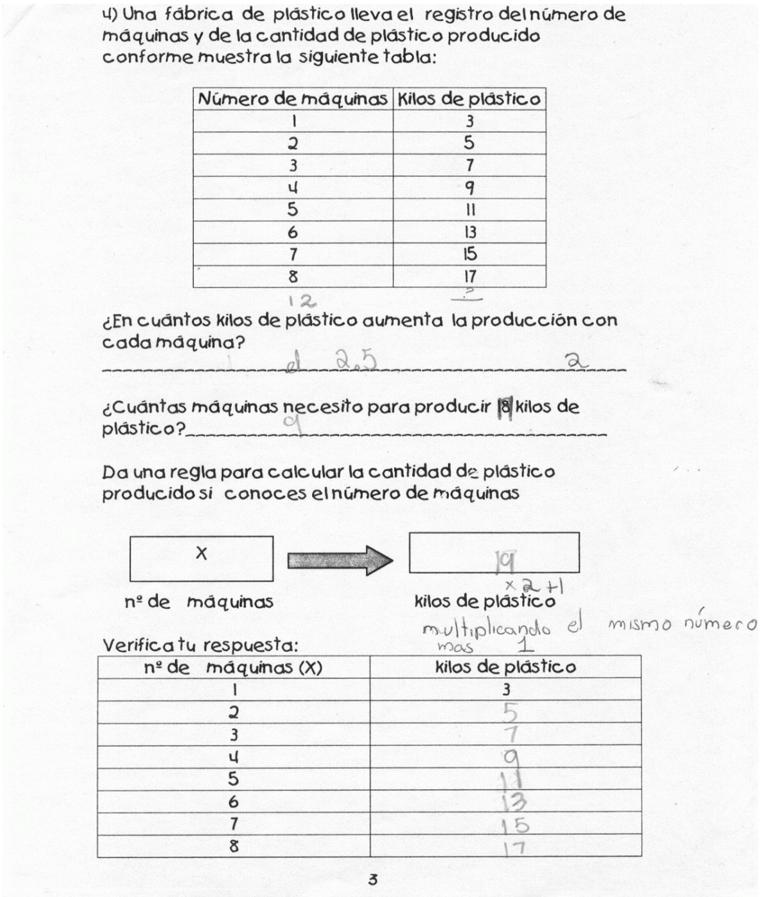
que es la 6ª parte

que aumentan las 2 columnas por su doble

Figure 2. A paper-and-pencil activity from the initial questionnaire: a mileage problem.

Questionnaire Problem 4

In the activity shown in Figure 3, children had to solve a word-problem dealing with a plastics factory that keeps records of machines and plastic quantities in kilograms (respectively, first and second columns of the tables). In this activity the linear relationship is explored. The children find the relationship among the number of kilograms of produced plastic and the number of machines involved, and they establish a pre-algebraic rule.



4) Una fábrica de plástico lleva el registro del número de máquinas y de la cantidad de plástico producido conforme muestra la siguiente tabla:

Número de máquinas	Kilos de plástico
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

12 2

¿En cuántos kilos de plástico aumenta la producción con cada máquina? el 2,5 2

¿Cuántas máquinas necesito para producir 18 kilos de plástico? 9

Da una regla para calcular la cantidad de plástico producido si conoces el número de máquinas

X → 19

nº de máquinas

kilos de plástico x2+1

multiplicando el mismo número mas 1

Verifica tu respuesta:

nº de máquinas (X)	kilos de plástico
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

3

Figure 3. Another paper-and-pencil activity from the initial questionnaire: the plastics factory.

Below is part of the translated transcript from the initial interview, after the interviewer reads out loud the problem and asks how it was solved:

Interviewer: [Reading the table] 2 machines, 5; 3 machines, 7; 4 machines, 9; 5 machines, 11; 6 machines 13; 7 machines, 15; 8 machines, 17.
By how many kilograms of plastic, does the production increases with each machine?

Child: 2 for each machine and then add 1.

Interviewer: How many machines do you need for producing 19 kilograms of plastic?

Child: 39.

Interviewer: If I want 19 kilograms of plastic, how many machines do I need?

Child: Te…, nine!

Interviewer: Why did you put here 39?

Child: I made a mistake. Nine!

Interviewer: What did you understand before?

Child: that it was the double and then you added 1 kilogram of plastic.

Interviewer: How did you find the rule?

Child: Because each machine produced 2 kilograms of plastic, plus 1, it's 3; that is, each machine added 1. That's why for 1 it's 3; for 2 it's 5; for 3 it's 7; for 4 it's 9, for 5 it's 12; for 6 it's 13; for 7 it's 15; for 8 it's 17.

Interviewer: Tell me, what does the rule say?

Child: For each machine multiply by 2 and add 1.

Sample data from the Logo activities

In the following figures, some sample work is shown from the Logo activities carried out with the students during the preliminary study. This work was done using WinLogo.

Figure 4a shows a Logo activity worksheet for drawing a letter 'N' (shown in Figure 4b) in different sizes. In filling out the table and comparing the commands[1] for drawing each letter 'N' of different size, the students need to observe if there is a something in common. The fact that students themselves construct each letter 'N' is very important in helping them see the relationships.

Ahora llena la tabla con los siguientes datos de cada letra " N"

Letras N	Medida del 1º segmento (PT)	Medida del 2º segmento (PT)	Medida del 3º segmento (PT)
nº 1	90	100	90
nº2	180	200	180
nº3	270	300	270
nº 4	450	500	450
nº 5	540	600	540

Compara los comandos de las letras "N" y responde si tienen algo en común se son multiplos de 10 y 100

Ahora, escribe los comandos de las Letras "N" y responde si tienen algo en común.

comandos para nº1

av 90
gd 150
av 100
gi 150
av 90

comandos para nº2

av 270
gd 150
av 300
gi 150
av 270

comandos para nº3

av 150
gd 150
av 300
gi 150
av 450

comandos para nº4

av 630
gd 150
av 700
gi 150
av 630

comandos para nº5

av 810
gd 150
av 900
gi 150
av 810

Figure 4a. Logo activity for drawing a letter 'N'

```
para n :lado
av :lado
gd 150
av :lado
gi 150
av :lado
fin
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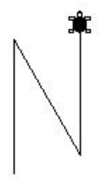

Figure 4b. Logo procedure for the letter 'N' with its result

[1] The Logo primitives are given in Spanish: AV is FD; GD is RT; GI is LT.

Another interesting activity is the one shown in Figure 5. In this activity generalisation processes are explored. The child is asked to observe a sequence of figures and then complete the figures for the 4th and 5th elements of the sequence. It is observed that the child completes the figures by counting the number of squares on the horizontal and vertical directions. Afterwards, the child fills a table of values and discovers a general rule for the figures. In the case of the child that filled the worksheet shown in Figure 5, he got the general rule for the case 11+10-1 and wrote how the number of H (for horizontal) and V (for vertical) squares are obtained from the figure number: H equals the 'figure number' plus 2; while V equals the 'figure number' plus 1.

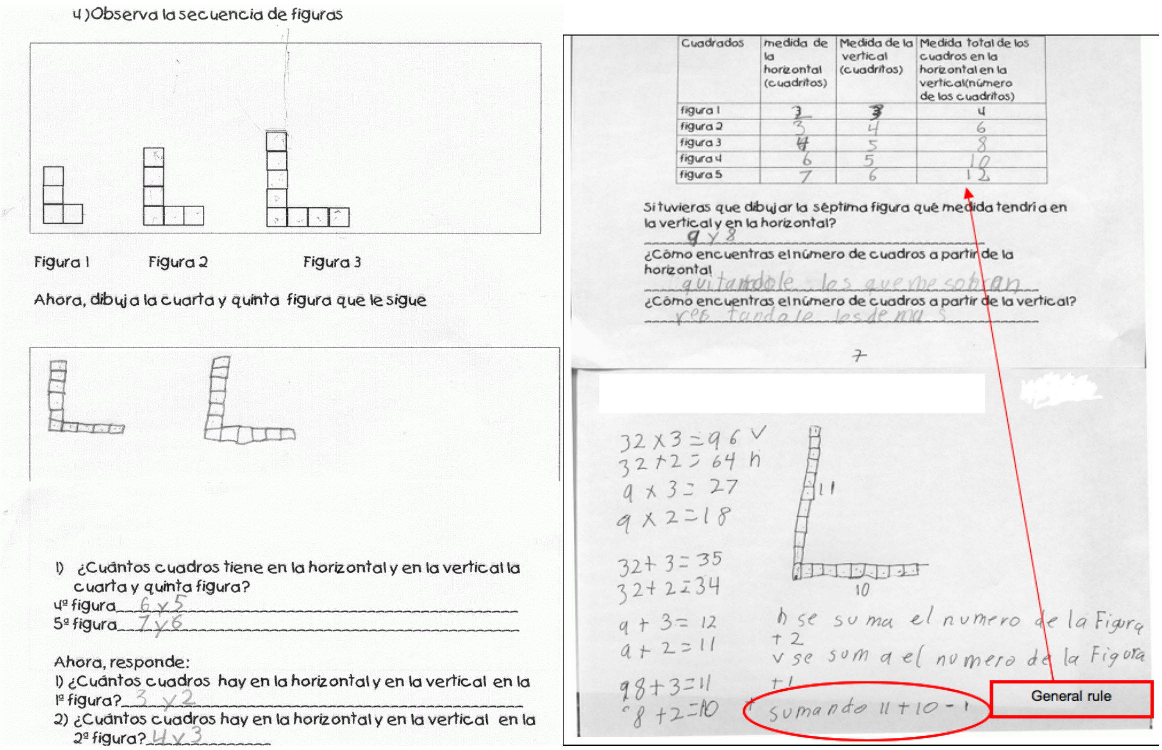

Figure 5. Pre-Logo activity in which a sequence of figures is observed and students have to reflect on what the general rule is that they would need to draw them in Logo

General results highlighting the changes between the beginning and end of the first phase

The results from the pre-questionnaire show that the participants were pre-algebraic students, which means that they didn't find difficult to understand some previous ideas, in spite of the difficulties that they found in the initial questionnaire. After the individual interview, it was confirmed that they reorganised their answers and were capable of recognizing their own mistakes as well as reorganise their thinking. In the pre-questionnaire, in problems that explore the idea of variable in a functional relationship, the students perceived the existent relationship among the problem quantities. They also perceived how the values of one of the variables increased and decreased, but they were not able to express this fact. They could express relationships between quantities in a table, but were unable to express them through a general rule. Instead they had to do this with a step-by-step description that did not allow them to generalise the relationship.

After the working sessions of the first phase, the children moved forward to more elaborate strategies in the resolution of the problems, showing conceptions of variable as a functional relationship and as general number. In the interview that was carried out after the working sessions, most of children changed the answers that they had given in the initial questionnaire. When the interviewer asked them to justify their answers, the children showed that they had advanced conceptually. For example, they were able to perceive a multiplicative relationship in problems involving a geometric sequence; in fact, two students perceived the functional relationship in data and were able to express a general rule. For the problems that approach the variable idea as general number, they understood, for example, how the quantities were varying, and seemed to attain an understanding of the relationships involved, even though in the paper-and-pencil environment some of them could not determine a general method and thus had problems in generating a general rule. However, most of the students were capable of verifying if a rule could function in all cases.

Final remarks

Introduction to early algebraic thinking along two routes of access (proportional reasoning and generalisation processes) – by means of a didactical sequence that takes into account the mathematical and cognitive background of the individuals – seems to be a viable approach and has correspondence with historical and curriculum perspectives. The results in the first stage of the study reveal some of the abilities and difficulties that are typical of the age-group we worked with.

With respect to the zone of actual development (ZDA), we verified that the interaction with the interviewer played a relevant role, because by this means, the students were able to make explicit the way they solved the problem as well as reconceptualise their knowledge. In some cases, help of the interviewer permitted that children could restructure their thinking by a simple solicitude of justification. In other cases, several levels of help were needed, depending on the real or actual evolutive level the children had. This real evolutive level (ZDA) could certainly be potentiated within appropriate technological environments for algebraic thinking, but also with a well structured design of activities from the didactical and psychological point of view.

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