Algebra and Computer Algebra: Implications for High School Mathematics Examples from *The CME Project*

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Abstract

This paper builds on more than a decade of work at Education Development Center on the use of computer algebra with high school teachers and students. Widespread CAS use is still in its infancy in US precollege education. Its acceptance into the high school curriculum has been hampered, both by its prohibition on several high-stakes exams (the American College Testing (ACT) exam, for example) and by worries among many high school and university faculty that its use will diminish students' technical fluency with algebraic calculations.[1] The situation in the EU seems to be quite different, and I hope that I can learn at this conference how this technology is being put to use and how it has gained acceptance in European high schools.

But in a real sense, the CAS and the technology, while essential tools in what follows, are not the foci of this paper or of the conference. One of the goals of *Constructionism 2010* is to encourage "learners to better understand the world and their place in it by building their own meaning-making models based on iterative, interactive exploration and testing of ideas and notions." I want to focus on using this process as it applies to one corner of the world of mathematics.

In this paper, I'll look at several examples of how CAS environments can be used to model algebraic systems and objects. The advantages of computer algebra over other programming languages come from the fact that, in CAS environments, algebraic expressions are first-class objects. Because formal expressions are (in a sense that can be made precise) universal objects for building algebraic structures, models and experiments built in these media realize two of the goals of the constructionist approach that go back to the early days of Logo: such models and experiments are both are *general-purpose* and *extensible*.

Equipped with working computational models of algebraic systems, many high school students and teachers can gain first-hand experience with the ideas that led naturally to modern abstract algebra, providing one more example of what Richard Noss states in his open letter: "...that the Logo vision could catalyse a transformation—not just of the *ways* that people learn, or of the *methods* by which they are taught—but of *what* it becomes possible to teach and learn."

Most of my examples are taken from *CME Project*, a high school curriculum, funded by the National Science Foundation, and published by Pearson in 2009 [1]. Details about the program are at `www.edc.org/cmeproject`. The features of *CME Project* that are relevant to this paper are

1. The program is organized around mathematical habits of mind [3].
2. It makes essential use of a CAS in the last two years.

Keywords

Computer algebra system, algebraic thinking

[1] Indeed, in the US, Texas Instruments has had to produce two versions of its new handheld—one with a CAS and a lobotomized version without computer algebra.

Introduction

One[2] can build meaning-making models of phenomena in a whole host of media; I watch my grandson create models of all kinds of things with his Lego blocks, and I see him create his imaginary worlds, full of interesting characters, with his Wii. But what about mathematics? One can certainly use mathematics to model physical phenomena, but how can one model the phenomena of mathematics itself? Many years ago, my colleague Paul Goldenberg and I put it this way:

> *But mathematical objects are objects of the imagination, and many . . . don't have physical models. How, then, can people tinker with these systems or the mathematical objects of which they're built? They may not have physical models, but they* **do** *have computational models. Algebraic structures, functions, continuously varying systems, and combinatorial enumerations can all be modeled in computational environments. When students build computational models of mathematical structures—whether these are programming models in languages like Logo, ISETL, or Mathematica, templates for a spreadsheet, or constructions in tools like* Geometer's Sketchpad *or* Cabri*—they are reviewing, expressing, and getting a chance to examine the own ideas about these mathematical structures. At one level, they are getting the benefit that generally comes from writing out one's ideas carefully and in detail: that process, by itself, helps one organize one's thinking, and externalize it enough to review and examine it. Without computational technology, students had to be satisfied with their written notes. The students who could bring these notes to life entirely in their heads would have more success than those for whom the notes just sat motionless on the paper. But when the "notes" are executable on a computer, students can run the models they've made, verify their correctness or debug them, and even use them as parts of more complex models. Students who are not yet skilled enough to hold many parts of a model in their heads can build the parts one by one, show how they go together and, for the present, leave the orchestration to the computer. In short, computers can help students tinker with the physics of mathematics.* [8]

The habits of mind approach

About 40 years ago, early in my high school teaching career, I came to understand that the real utility of mathematics for many students comes from the kind of thinking that is indigenous to the discipline. In [2], I put it this way:

> *I didn't always feel this way about mathematics. When I started teaching high school, I thought that mathematics was an ever-growing body of knowledge. Algebra was about equations, geometry was about space, arithmetic was about numbers; every branch of mathematics was about some particular mathematical objects. Gradually, I began to realize that what my students (some of them, anyway) were really taking away from my classes was a* style of work *that manifested itself between the lines in our discussions about triangles and polynomials and sample spaces. I began to see my discipline not only as a collection of results and conjectures, but also as a collection of* habits of mind*.*
>
> *This focus on mathematical ways of thinking has been the emphasis in my classes and curriculum writing ever since, and I'm now convinced that, more than any specific*

[2] I'd like to thank Wally Feurzeig, both for his help with this paper and for all that he's done for mathematics education over the years.

> *result or skill, more than the Pythagorean theorem or the fundamental theorem of algebra, these mathematical habits of mind are the most important things students can take away from their mathematics education. For* all *students, whether they eventually build houses, run businesses, use spreadsheets, or prove theorems, the real utility of mathematics is not that you can use it to figure the slope of a wheelchair ramp, but that it provides you with the intellectual schemata necessary to make sense of a world in which the products of mathematical thinking are increasingly pervasive in almost every walk of life.*

When I first came to EDC in the early 1990s, my colleagues and I made a careful analysis of these mathematical habits of mind (see [3], for example), and we began developing high school courses and curricula organized around this analysis. *CME Project* is a direct descendent of that early work and the decades of classroom experience that preceded it; the evolution is described in more detail in [5].

By "mathematical habits of mind," I mean the mental habits that mathematicians use, often unconsciously, in their mathematical work. There are general mathematical habits—performing thought experiments, for example—and habits that are central to specific branches of mathematics. In analysis, for example, one often employs reasoning by continuity or passing to the limit. There are also important *algebraic* habits of mind that are the focus of the algebra courses in *CME Project*. These include:

- Seeking regularity in repeated calculations.
- "Chunking" (changing variables in order to hide complexity).
- Reasoning about and picturing calculations and operations.
- Purposefully transforming and interpreting expressions to reveal hidden meaning.
- Seeking and modeling structural similarities in algebraic systems.

Developing these and related algebraic habits is a pervasive goal throughout the program. So, for example, *CME Project* develops an approach to solving classical algebra word problems, not because of any intrinsic value in these problems and their stylized contexts, but because this class of problems, and the approach students use to solve them, provides an arena for developing the extremely useful habit of finding regularity in repeated calculations and forming processes from isolated computations.

Our choices of technologies and how we use them is also dictated by this goal of developing specific mathematical habits. For example, dynamic geometry environments can be used to help students learn to reason by continuity and to look for invariants under continuous transformations.

Computer algebra systems are ideal media for helping students develop algebraic habits like the ones described above. And access to a CAS gives students much more than computational power and the ability to perform complicated calculations.

Using a CAS to build algebraic habits of mind

Modern CAS environments contain a great deal more than the ability to treat algebraic expressions as first-class objects (that is, objects that can be named and that can be inputs to and outputs from functions). The TI-Nspire technology, for example, has graphics-handling capabilities (including equation graphing and dynamic geometry), a spreadsheet, a functional programming language, and a CAS, and all of these environments talk to each other. We make use of all of these capabilities in *CME Project*, but I want to focus here on the value-added that comes from computer algebra: the ability to use these packages with formal algebraic expressions.

Our group at EDC sees three overlapping uses for computer algebra that help students develop algebraic habits: CAS media can be used as

an algebra laboratory. CAS technology can be used to experiment with algebraic expressions in the same way that calculators can be used to experiment with numbers: generating data, making patterns apparent, and giving students the raw data from which they can generate conjectures. They provide teachers and students with general purpose tools for finding regularity in data, or for imposing regularity when no simple patterns can be found. CAS technology also has the potential to bring a renewed and modern emphasis on formal algebra—that is, the algebra of forms—to school mathematics (see [6] and [7] for more on this theme).

an algebraic calculator. CAS technology can be used to make tractable and to enhance many beautiful classical topics, historically considered too technical for high school students. This is the use of technology that reduces computational overhead and that allows students to easily perform calculations that would be impossible (or overly distracting) without the technology. It is also the use that surrounds one of the biggest worries of many teachers in the US: If the computer can perform the calculations, what is the value of teaching paper-and-pencil algebraic skills?

a modeling tool for algebraic structures. This is the use that's of most importance to constructionism. CAS technology allows students to build models of algebraic objects and systems that have no faithful physical counterparts. This use of technology adheres to our view that building a computational model for a mathematical structure helps one build the mental constructions needed to interiorize that structure [8, 10]. Furthermore, such computational models are *executable*, so that students can build working models of mathematical systems, turning the mathematician's thought experiments into actual experiments. As we said on page 2, what CAS environments add to other modeling environments is the facility to perform generic calculations with algebraic *expressions*—polynomials, rational functions, and formal power series. Hence these environments provide a medium for expressing abstract algebraic structure.

Of all the computational available environments, the TI-Nspire system is best suited for our purposes for several reasons:

1. It is first and foremost an educational tool, so that great care has gone into the design of its interface and its conventions. For example, it uses notation that is faithful to common mathematical notation—what you write on the blackboard is essentially what you type into the system.
2. It is available on a handheld device, so that students can use the system in or out of class.
3. The various other environments (dynamic geometry, functional programming, and spreadsheet, for example) are also designed for education, and the various environments interact. So, for example, a function defined in the programming environment can be tabulated in the spreadsheet.

Examples: A case study of $x^n - 1$

In this section, I'll look at each of the CAS uses described above—experimenting, calculating, and modeling—pointing out how they encourage the development of algebraic habits.

The context for these examples is the set of polynomials of the form $x^n - 1$, where n is a positive integer. These polynomials are ubiquitous in almost every branch of mathematics. From a high school curriculum perspective, they can be used to tie together many core results from algebra, geometry, and trigonometry. My goal in these examples is to show how CAS models of the mathematical objects help reify the objects in the minds of people who build the models. Twenty minutes after bringing the next example into a classroom or a workshop, there's no question about the fact that everyone feels that they are dealing with real objects.

Experimenting: Finding factors of $x^n - 1$

Most first-year algebra books contain the factorization

$$x^2 - 1 = (x-1)(x+1)$$

Sometime in high school, students may also see

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1) \\ x^4 - 1 &= (x-1)(x+1)(x^2+1) \\ x^6 - 1 &= (x-1)(x+1)(x^2+x+1)(x^2-x+1) \end{aligned}$$

So, over the integers $\mathbb{Z}$, $x^2 - 1$ and $x^3 - 1$ each have two factors, $x^4 - 1$ has three, and $x^6 - 1$ has four. Is there any pattern to the number of factors as a function of n? That is, can we find any regularity in this table?

n	number of factors of $x^n - 1$
1	1
2	2
3	2
4	3
5	
6	4
7	
8	
9	

A CAS allows one to experiment with this question, generating data from which one can draw conclusions. For example, you can define a function that factors the polynomials:

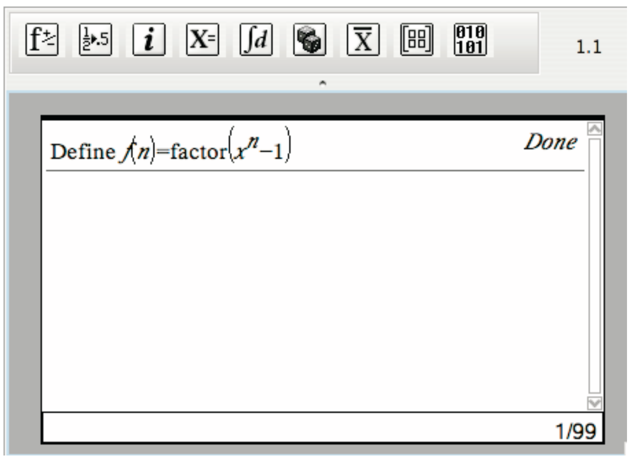
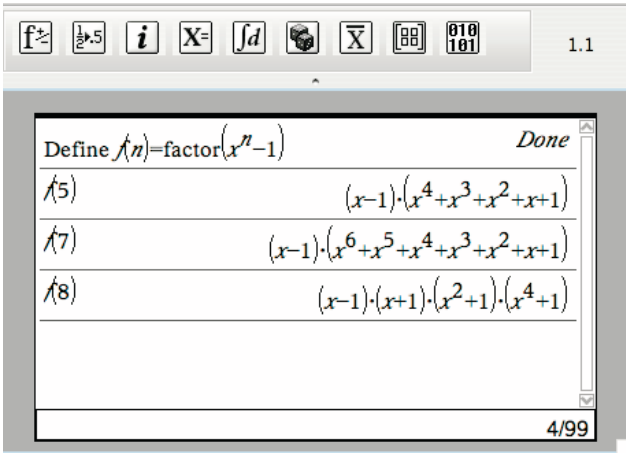


Figure 1: $f(n)$ factors $x^n - 1$ over $\mathbb{Z}$

The experiment might proceed as follows



At this point, two conjectures often emerge:

1. There are always at least two factors:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)$$

2. If n is odd, there are exactly two factors.

The first conjecture is true; the factor theorem from algebra 2 shows that $x-1$ must be a factor of $x^n - 1$ for any n, because 1 is a root of the equation $x^n - 1 = 0$. In *CME Project*, we ask students to explain why the right-hand side multiplies out to $x^n - 1$ *without carrying out any explicit calculations*, picturing how the the calculation would go if they did multiply everything out.

Conjecture 2 is false, as a little more experimenting shows:

n	number of factors of $x^n - 1$
1	1
2	2
3	2
4	2
5	2
6	4
7	2
8	4
9	3

When we've used this activity with students and teachers, several conjectures emerge:

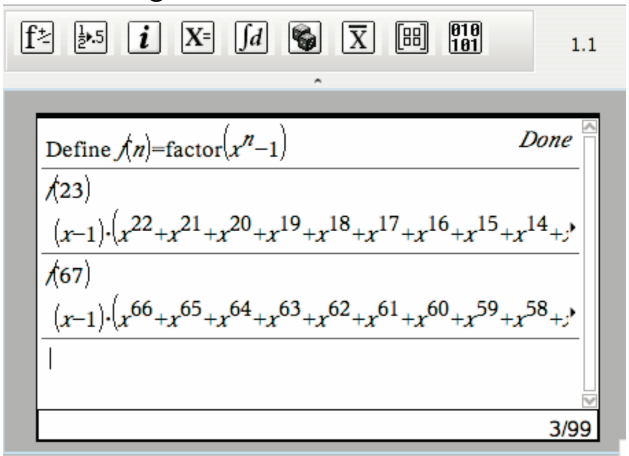
- If n is *prime*, there are exactly two factors.
- If n is the square of a prime, there are three factors ($x^9 - 1$, for example).
- If n is the product of two distinct primes, there are four factors ($x^{15} - 1$, for example).

In classroom discussions or in student work, these statements usually coalesce into a single conjecture:

> **Conjecture**: *The number of irreducible factors of $x^n - 1$ over $\mathbb{Z}$ is the number of positive integer factors of n.*

Here we have a conjecture for a non-obvious (and non-trivial) pattern in a sequence of polynomials. When I've used this activity with students and teachers, the question takes on a life of its own, and the laboratory environment afforded by the CAS helps establish the claim I made on page 4: the objects of the investigation (the polynomials) become *real objects*. Some other points about this investigation:

- The CAS can be used to check conjectures for large values of n, adding to the sense that one is working with real "things:"



- By looking at the actual factorizations produced by the CAS, rather than simply the number of factors, one can develop and prove more refined results. Indeed, the CAS can be used to inspire results about the factorizations of certain subsets of our sequence:

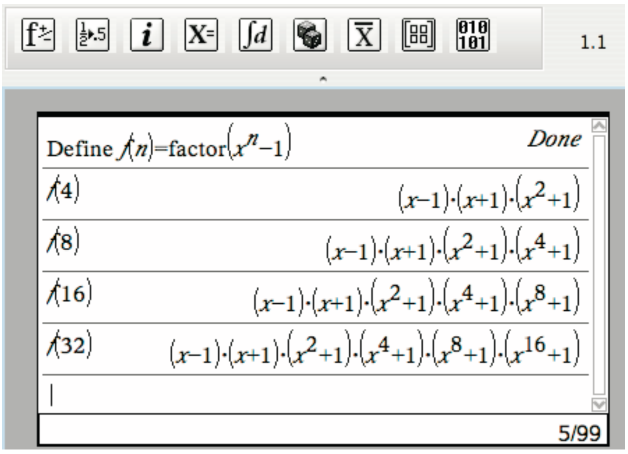


Figure 2: $f(2^k)$ as a special case.

- CAS use makes progress on a conjecture tractable for almost all second-year algebra students, and may of them will leave it at that. Others may take things a bit further and show why $\frac{x^n-1}{x-1}$ is irreducible if n is prime.

This is a good example of a low-threshold, high-ceiling activity. And the mathematics behind all this is central to many parts of algebra and analysis—it gets deep enough to challenge even the most advanced students. For example, if $\psi_k(x)$ is the polynomial whose roots are precisely the primitive kth roots of unity, then

$$x^n - 1 = \prod_{d|n} \psi_d(x) \qquad (*)$$

Here, the product is over all divisors of n. It can be shown (although the standard proof is quite hard in places) that each $\psi_k(x)$ is defined and irreducible over $\mathbb{Z}$, explaining why the conjecture on page 6 is, in fact, true. And equation $(*)$ can be used in a CAS to compute each $\psi_k(x)$ recursively.

More refined conjectures emerge from further experimentation. Whenever I use this activity with students or teachers, someone always asks if the coefficients of the $\psi_k(x)$ are always in the set

$$\{0, \pm 1\}$$

One can use a CAS to investigate this question. The first instance of a coefficient different from $0, \pm 1$ is in ψ_{105}. In fact, the coefficients of ψ_n can be made as large as one pleases [9]. There's much more to say about this example, but the point here is that we are now dealing with genuine models of real phenomena, with all the textured features of intricate physical systems.

Reducing overhead: The Polynomial Factor Game

The *Connected Mathematics Project* [11] introduces middle school students (ages 11–13) to primes and the prime factorization of integers via the *Factor Game*. This is a game for two players, played on a board like this:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

The rules of the game are up for negotiation in a class, but one version goes like this:

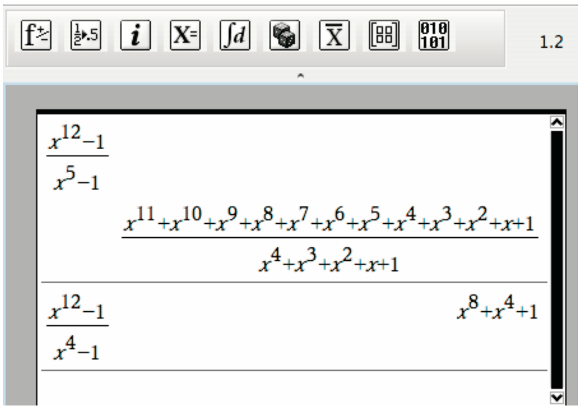
1. Player A picks a number n from the board, getting that many points, and the number is crossed off.
2. Player B gets the sum of all the numbers not crossed off on the board that are factors of n, and crosses them off.
3. B goes next, picking an available number and gets that value.
4. A gets the sum of the non-crossed off numbers that are factors of m.
5. If either player picks a number with no factors left on the board, he or she loses a turn and gets no points.
6. The game continues until there are no possible moves.

CME Project contains a game with the same rules, except the board looks like this:

$x-1$	x^2-1	x^3-1	x^4-1	x^5-1
x^6-1	x^7-1	x^8-1	x^9-1	$x^{10}-1$
$x^{11}-1$	$x^{12}-1$	$x^{13}-1$	$x^{14}-1$	$x^{15}-1$
$x^{16}-1$	$x^{17}-1$	$x^{18}-1$	$x^{19}-1$	$x^{20}-1$
$x^{21}-1$	$x^{22}-1$	$x^{23}-1$	$x^{24}-1$	$x^{25}-1$
$x^{26}-1$	$x^{27}-1$	$x^{28}-1$	$x^{29}-1$	$x^{30}-1$

The points that a player wins on a round correspond to the degrees of the polynomials that are picked.

The CAS is used here simply as an algebraic calculator. If a player wants to see if one of these polynomials divides another, he or she can simply check to see if the quotient is a polynomial.



It doesn't take long before students begin to see that this game "is the same as the middle school factor game." That is, a conjecture emerges

Conjecture: $x^m - 1$ is a factor of $x^n - 1 \Leftrightarrow m$ is a factor of n

One direction of this implication is a nice application of the "chunking" habit: To see, for example, that $x^3 - 1$ is a factor of $x^4 - 1$, you can argue like this:

$$\begin{aligned} x^{12} - 1 &= \left(x^3\right)^4 - 1 \\ &= (\clubsuit)^4 - 1 \\ &= (\clubsuit - 1)\left(\clubsuit^3 + \clubsuit^2 + \clubsuit + 1\right) \quad \text{(see the identity on page 6)} \\ &= \left(x^3 - 1\right)\left((x^3)^3 + (x^3)^2 + (x^3) + 1\right) \\ &= \left(x^3 - 1\right)\left(x^9 + x^6 + x^3 + 1\right) \end{aligned}$$

The other direction of the implication (if $x^m - 1$ is a factor of $x^n - 1$, m is a factor of n) is much harder. One way to think about it it requires some facility with De Moivre's theorem and with roots of unity. Another approach (shown to me byVince Matsko) is to use the arithmetic structure of the ring of polynomials in one variable over the real numbers, a structure with many of the same features as the ring of ordinary integers. Briefly, it goes like this:

Suppose that $x^m - 1$ is a factor of $x^n - 1$. Write $n = mq + r$ with $0 \le r < m$. Then

$$x^{n-r} - 1 = x^{qm} - 1$$

But $x^m - 1$ is a factor of the right-hand side of this equation (chunking, again), so it divides both $x^n - 1$ and $x^{n-r} - 1$, and hence divides their difference:

$$x^{n-r}(x^r - 1)$$

But $x^m - 1$ is relatively prime to x^{n-r}, so it must be a factor of $x^r - 1$. Since $r < m$, this implies that $r = 0$.

Modeling: Roots of unity

If you watch high school students calculate with complex numbers, many will act as if they are calculating with polynomials in i, with the additional simplification rule "$i^2 = -1$." There is a germ of an important idea here: students are noticing the structural similarities between $\mathbb{C}$ and

$\mathbb{R}[x]$—the two systems seem to "calculate the same." This is a good example of the universal nature of formal algebraic expressions mentioned on page 1: The complex numbers can be realized as a "quotient" of $\mathbb{R}[x]$ by the relation $x^2 + 1 = 0$ (see [4] for more on this theme). And in fact this construction, first articulated in this way by Kronecker, is perfectly general: every algebraic extension of a field K can be modeled as $K[x]$ with some extra relations.

This seeking structural similarities in algebraic systems is an important algebraic habit of mind, and it gets exercised when calculations in one system start to feel like calculations in another. But before the advent of CAS, I would have never thought of introducing it to any but the most advanced precollege students. Now it becomes, without all the trappings of abstract algebra, tractable to a wider set of students and teachers.

For example, many precalculus courses (including *CME Project*) contain a treatment of De Moivre's theorem, often stated like this:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

De Moivre's Theorem implies several facts relevant to our family $x^n - 1$:

- The roots of $x^n - 1 = 0$ are

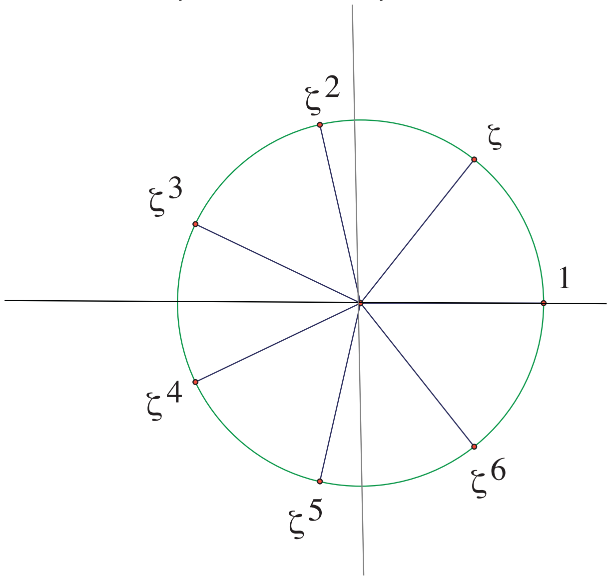
$$\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad | \quad 0 \le k < n\right\}$$

- If $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, these roots are

$$1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{n-1}$$

- These roots lie on the vertices of a regular n-gon of radius 1 in the complex plane.

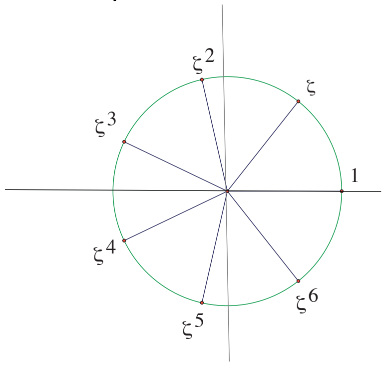
In *CME Project* precalculus book, an optional suite of problems deals with the 7th roots of unity:



Notice that

- The six non-real roots come in conjugate pairs.
- So, $(\zeta + \zeta^6)$, $(\zeta^2 + \zeta^5)$, and $(\zeta^3 + \zeta^4)$ are real numbers.
- Hence these three numbers *satisfy a cubic equation over* $\mathbb{R}$.

The object of the activity is to find this equation.



Let

$$\alpha = \zeta + \zeta^6$$
$$\beta = \zeta^2 + \zeta^5$$
$$\gamma = \zeta^3 + \zeta^4$$

To find an equation satisfied by α, β, and γ, we need to find

- $\alpha + \beta + \gamma$
- $\alpha\beta + \alpha\gamma + \beta\gamma$
- $\alpha\beta\gamma$

We find these one at a time...

The Sum:

Since $\alpha = \zeta + \zeta^6$, $\beta = \zeta^2 + \zeta^5$, and $\gamma = \zeta^3 + \zeta^4$, we have

$$\alpha + \beta + \gamma = \zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta$$

But

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

So,

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta = -1$$

The Product:

$$\alpha\beta\gamma = \left(\zeta + \zeta^6\right)\left(\zeta^2 + \zeta^5\right)\left(\zeta^3 + \zeta^4\right)$$

Notice that the right-hand side "feels like" a call to do a formal calculation. Indeed, we can get the *form* of the expansion by expanding

$$\left(x + x^6\right)\left(x^2 + x^5\right)\left(x^3 + x^4\right)$$

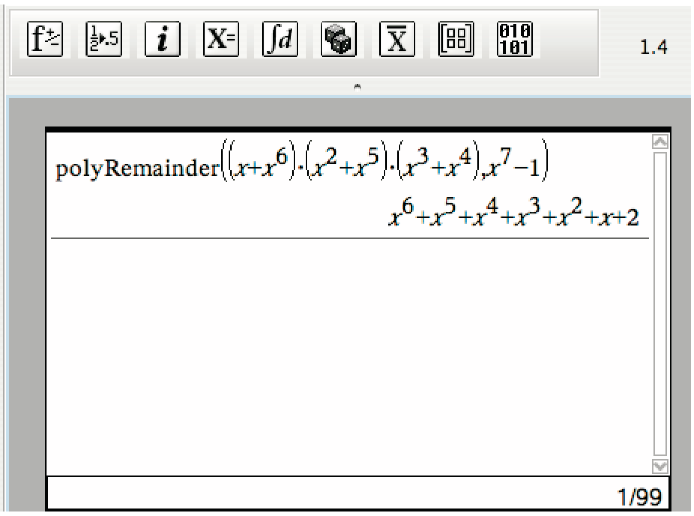
A CAS tells us that

$$\left(x + x^6\right)\left(x^2 + x^5\right)\left(x^3 + x^4\right) =$$
$$x^{15} + x^{14} + x^{12} + x^{11} + x^{10} + x^9 + x^7 + x^6$$

But if we replace x by ζ, we can replace x^7 by 1. So, if the above expression is divided by $x^7 - 1$ and written as

$$(x^7 - 1)q(x) + r(x),$$

then replacing x by ζ will produce $r(\zeta)$. A CAS can be used to do the calculation:



Since

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$$

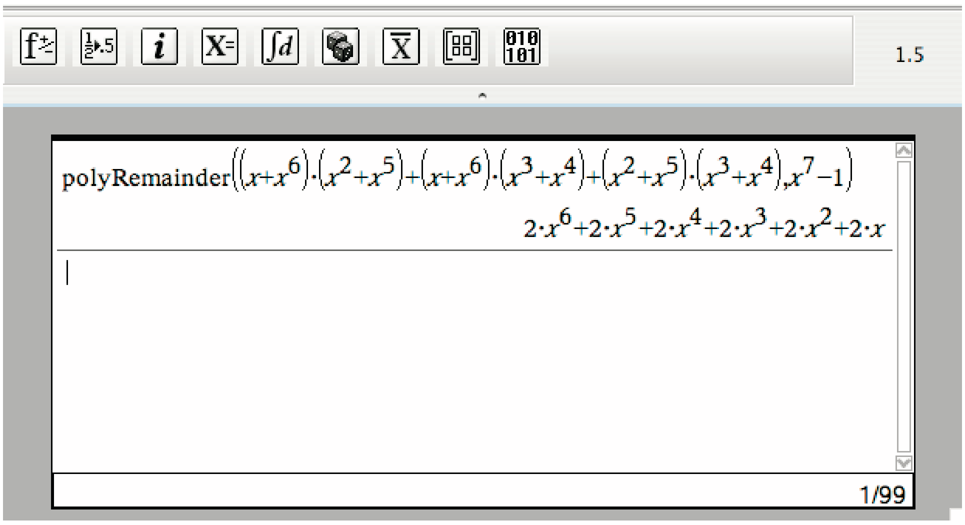
We get

$$\alpha\beta\gamma = 1$$

The sum, two at a time: Well, $\alpha\beta + \alpha\gamma + \beta\gamma =$

$$\begin{gathered}\left(\zeta+\zeta^6\right)\left(\zeta^2+\zeta^5\right)+\\\left(\zeta+\zeta^6\right)\left(\zeta^3+\zeta^4\right)+\\\left(\zeta^2+\zeta^5\right)\left(\zeta^3+\zeta^4\right)\end{gathered}$$

We can use a CAS, thinking of this as a formal calculation, reducing by $x^7 - 1$:



It follows that $\alpha\beta + \alpha\gamma + \beta\gamma = -2$, and our cubic is

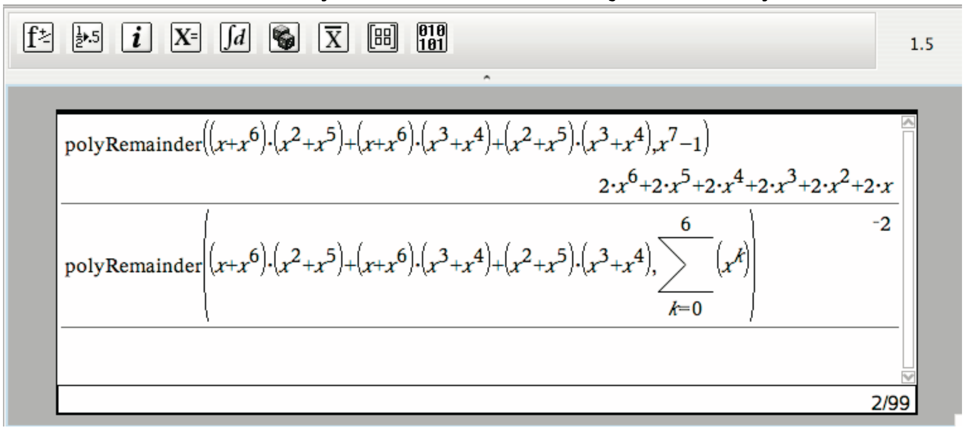
$$x^3 + x^2 - 2x - 1 = 0$$

There are several purposes for this exercise in addition to giving a concrete (computational) preview Kronecker's construction of splitting fields for algebraic equations:

- In an informal way, students preview the idea that one can model $\mathbb{Q}(\zeta)$ by "remainder arithmetic" in $\mathbb{Q}[x]$, using $x^7 - 1$ as a divisor.
- In fact, one can use any polynomial that has ζ as a zero—the smallest degree one is

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

  Doing so would have reduced significantly the simplifications needed at the end of each step, and the CAS would carry out the calculations just as easily.



- The CAS model allows students to experiment with arithmetic in $\mathbb{Q}(\zeta)$ by performing arithmetic with polynomials.

Veteran Logo users will recognize that this idea of of modeling algebraic structures goes back to, for example, the Logo activities in which students modeled $\mathbb{C}$ via arithmetic with paris of real numbers. The version presented here is a kind of refinement of those ideas, this time using formal algebraic expressions as the modeling tool rather than data structures like lists.

On CAS Use

CAS environments have been used for over a decade in undergraduate mathematics, and now, with the availability of these media on handheld devices, they are gradually making their way into precollege (upper secondary) programs. Especially in the United States, where jumping on bandwagons has a longstanding and quasi-respectable tradition in education, two opposing camps are developing:

- Many people are worried that the influx of CAS environments into precollege mathematics will produce a generation of high school students who reach for a calculator to factor x^2+x, much like the alleged current generation of college students who reach for a calculator to multiply 57 by 10.
- And there are those who adopt the motto "if the machine can do it, why bother teaching it?"—many educators are proclaiming that facility with algebraic calculation is unnecessary and that we can do away with those tortuous pages of factoring, simplifying, and solving.[3]

Experience tells us that both of these extreme stances will evolve eventually into something much less grandiose and that CAS environments will take their place alongside other useful computational media as enhancements to, rather than replacements for, the essential role that technical fluency plays in mathematical understanding.

[3]Paul Goldenberg was at a meeting of US mathematics curriculum developers some years ago when someone made the comment that algebra is dead, causing a roaring round of applause from the audience.

In this paper, I've provided one example of how CAS environments can be used to enhance the high school algebra curriculum. *CME Project* uses CAS technology to

1. *Experiment with algebra*
2. *Reduce computational overhead*
3. *Use polynomials as modeling tools*

Our work with teachers and students thus far has convinced me that computer algebra is a very useful tool to help people bring the objects of mathematics, especially formal mathematical expressions, into their realities.

The examples given in the previous sections are just that: examples. There are many other examples of modeling opportunities that have little to do with $x^n - 1$: Chebyshev polynomials, Lagrange interpolation, Newton's difference formula, and generating functions, just to name a few. All of this beautiful and classical mathematics is now accessible to many more students than in previous decades, and all of it becomes "real" in a CAS environment.

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