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An abstract theory of topological subspaces

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1. *Introduction*. The open subsets of a topological space X form a complete Brouwerian, i.e. distributive and pseudo-complemented lattice $L(X)$. Many topological properties of X can be formulated as properties of $L(X)$, although, in general, X is not determined by $L(X)$. Obvious examples are quasi-compactness and connectivity: X is quasi-compact if any set of elements of $L(X)$ whose join is I has a finite subset whose join is I; X is connected if no element of $L(X)$ has a complement. These properties which we shall call 'paratopological', can be defined for lattices that are not of the form $L(X)$, and many topological theorems can be proved in this more general context. The purpose of this paper is to develop sections of general topology as part of the theory of complete Brouwerian lattices (CBL).

2. *Notations and some general properties of CBL*. Let L and K be complete lattices. A homomorphism $f: L \to K$ will be called a V-homomorphism if $f(Vx_a) = Vf(x_a)$ for arbitrary sets $\{x_a\}$ of elements of L. Similarly an equivalence relation R will be called a V-equivalence relation if $x_a R y_a$ implies $(Vx_a) R (Vy_a)$. L is a V-sublattice of K if it is a sublattice closed under arbitrary K-joins. Thus, the lattice of open sets $L(X)$ is a V-sublattice of the lattice of all subsets of the abstract set X, and the inverse map $f^{-1}: L(Y) \to L(X)$ induced by a continuous map $f: X \to Y$ is a V-homomorphism.

Let R be an equivalence relation on a lattice L and $\hat{x}$ the equivalence class containing the element x. We recall that the set of equivalence classes is a lattice, L/R, with respect to the order defined by

$$\hat{x} < \hat{y} \Leftrightarrow (Ex')(Ey')(x'Rx \,\&\, y'Ry \,\&\, x' < y').$$

Let a be an element of a CBL, L, and define the relation R_a by

$$xR_a y \Leftrightarrow a \wedge x = a \wedge y.$$

Proposition 1. *If L is a CBL then*

(*a*) *the relation R_a is a V-equivalence relation*;
(*b*) *the canonical map $f: L \to L/R_a$ is a V-homomorphism*;
(*c*) *the quotient L/R_a is a CBL.*

Proof. (*a*) xR_a and $x'R_a y'$ imply $x \wedge a = y \wedge a$ and $x' \wedge a = y' \wedge a$, so that

$$x \wedge x' \wedge a = y \wedge y' \wedge a,$$

i.e. $(x \wedge x')\,R(y \wedge y')$. Suppose that for each α in some set of indices $(x_\alpha)R_a(y_\alpha)$, i.e. $x_\alpha \wedge a = y_\alpha \wedge a$. This implies $V(x_\alpha \wedge a) = V(y_\alpha \wedge a)$ and, by the infinite distributivity of CBL, $a \wedge Vx_\alpha = a \wedge Vy_\alpha$, i.e. $(Vx_\alpha)\,R_a(Vy_\alpha)$.

(*b*) Let $x = Vx_\alpha$. We shall show that $\hat{x} = V\hat{x}_\alpha$. Clearly $\hat{x}$ is an upper bound to the $\hat{x}_\alpha$. Suppose that $\hat{y}$ is another upper bound, i.e. $\hat{y} > \hat{x}_\alpha$ for each α. This means that there exist y' and x'_α such that $y' \wedge a = y \wedge a$, $x'_\alpha \wedge a = x_\alpha \wedge a$ and $y' > x'_\alpha$. But

$$y' > x'_\alpha \Rightarrow y' \wedge x'_\alpha = x'_\alpha \Rightarrow y' \wedge x'_\alpha \wedge a = x'_\alpha \wedge a \Rightarrow y \wedge x_\alpha \wedge a = x_\alpha \wedge a.$$

By the infinite distributivity law it follows that $y \wedge x \wedge a = x \wedge a$, whence $(y \wedge x)^\wedge = \hat{x}$, i.e. $\hat{y} \wedge \hat{x} = \hat{x}$, i.e. $\hat{y} > \hat{x}$. Thus $\hat{x}$ is the l.u.b. of the $\hat{x}_\alpha$, i.e. $\hat{x} = V\hat{x}_\alpha$.

This proves that $f(x) = \hat{x}$ is a V-homomorphism.

(*c*) That L/R_a is a CBL follows immediately since

$$\begin{aligned} V(\hat{z} \wedge \hat{x}_\alpha) &= V(f(z) \wedge f(x_\alpha)) \\ &= f(V(z \wedge x_\alpha)) \\ &= f(z \wedge Vx_\alpha) \\ &= f(z \wedge Vx_\alpha) \\ &= f(z) \wedge Vf(x_\alpha). \end{aligned}$$

We shall in the sequel denote by $C(a)$ the set $\{x \in L;\ x < a\}$ and by $L/C(a)$ the quotient L/R_a.

We shall make special use also of quotients L/R'_α where $xR'_a y \Leftrightarrow x \vee a = y \vee a$.

PROPOSITION 1′.

(*a*) *R'_α is a V-equivalence relation;*

(*b*) *The canonical map $f: L \to L/R'_\alpha$ is a V-homomorphism;*

(*c*) *L/R'_α is a CBL.*

Proof. (*a*) $x \vee a = y \vee a$ and $x' \vee a = y' \vee a$ imply

$$(x \wedge x') \vee a = (x \vee a) \wedge (x' \vee a) = (y \vee a) \wedge (y' \vee a) = (y \wedge y') \vee a,$$

so that $xR'_\alpha y$ and $x'R'_\alpha y'$ imply $(x \wedge x')\,R'_\alpha(y \wedge y')$. Now suppose that, for each α in some set of indices, $x_a R'_\alpha y_\alpha$, i.e. $x_\alpha \vee a = y_\alpha \wedge a$. Clearly $V(x_\alpha \vee a) = V(y_\alpha \vee a)$, so that

$$(Vx_\alpha) \vee a = (Vy_\alpha) \vee a,$$

i.e. $(Vx_\alpha)\,R'_\alpha(Vy_\alpha)$.

(*b*) Let $x = Vx_\alpha$. It is obvious that $\hat{x}$ is an upper bound to the $\hat{x}_\alpha$. Suppose $\hat{y}$ is another upper bound. Then there exist elements $\hat{y}'_\alpha$ and x'_α such that $y'_\alpha \vee a = y \vee a$, $x'_\alpha \vee a = x_\alpha \vee a$ and $u'_\alpha > x'_\alpha$ for each α. But then $y'_\alpha \vee x'_\alpha = y'_\alpha$, so that $y'_\alpha \vee x'_\alpha \vee a = y' \vee a$. It follows that $y \vee x_\alpha \vee a = y \vee a$ and so $y \vee x \vee a = y \vee a$, i.e. $(y \vee x) \in \hat{y}$, so that $\hat{y} > \hat{x}$. This means that $\hat{x}$ is the l.u.b. of the $\hat{x}_\alpha$, i.e. $(Vx_\alpha)^\wedge = V\hat{x}_\alpha$.

(*c*) Exactly as for R_a.

3. *Subspaces.* Our example of quasi-compactness is an obvious case of a paratopological property. Less obvious examples can be based on a paratopological form of the property of being a subspace (more precisely: homeomorphic to a subspace) of a given topological space. (All topological spaces will be assumed to be T_0-spaces.) Let

Y be a subspace of X and $f: Y \to X$ be its injection map. Since the open sets of Y are the traces of open sets of X, the inverse map f^{-1} is epimorphic. The following proposition asserts the converse.

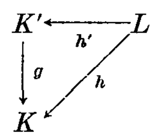
PROPOSITION 2. *If $f^{-1}: L(X) \to L(Y)$ is epimorphic, then Y is a subspace of X and its injection map.*

Proof. To prove that f is monomorphic choose two points, p and q of Y and suppose that there is an open set O, which contains p but not q. Since f^{-1} is epimorphic, there is an open set U such that $f^{-1}(U) = O$. But then $f(p) \neq f(q)$. Thus f embeds Y in X and it is easy to see that this embedding is homeomorphic.

Proposition 2 suggests the definition

DEFINITION 1. *A subspace of a CBL, L, is a couple (H, f) where H is a CBL and $f: L \to H$ is an epimorphic V-homomorphism.*

Let L be a fixed CBL and $S(L)$ the set of its subspaces. We order $S(L)$ by defining $(K, h) < (K', h')$ to mean that there exists an epimorphic V-homomorphism $g: K' \to K$ such that $h = gh'$, i.e.



PROPOSITION 3. *If (K, h) and (K', h') are identified whenever $(K, h) < (K', h')$* and *$(K', h') < (K, h)$, then $S(L)$ is a complete lattice under the ordering L. Moreover, $S(L)$ is identical with the complete lattice $R(L)$ of V-congruence relations on L.*

Proof. Suppose that $R \in R(L)$ and h is a canonical map $L \to L/R$. Then $(L/R, h)$ is a subspace of L. On the other hand if (K, f) is a subspace and xRy is defined by $f(x) = f(y)$, we can find an isomorphism g: $K \leftrightarrow L/R$ such that $gf = h$. Thus (K, f) and $(L/R, h)$ are identified in $S(L)$. Finally, the order of $S(L)$ coincides with the order of $R(L)$ since if R and R' correspond to K and K', $(K, h) < (K', h')$ if and only if $xR'y$ implies xRy, i.e. $R < R'$ in $R(L)$. The join of (R_a) in $R(L)$ is constructed by defining xRy by $xR_a y$ for all a. Hence the join in $S(L)$ of (K_a, h_a) is constructed by defining xRy as '$h_a(x) = h_a(y)$ for all a', and forming L/R. The join is then $(L/R, h)$.

DEFINITION 2. (*a*) *An open subspace of L is a subspace of the form $(C(x), i)$ where $C(x) = \{z: z \in L$ and $z < x\}$ and $i(z) = z \wedge x$.*

(*b*) *A closed subspace of L is a subspace of the form $(L/C(x), h)$ where $L/C(x)$ is the quotient of L by the V-congruence relation $u \equiv v \Leftrightarrow u \vee x = v \vee x$ and h is the canonical map $L \to L/C(x)$.*

PROPOSITION 4. *The map $\tilde{C}: L \to S(L)$ defined by $\tilde{C}(x) = (C(x), i)$ is a V-homomorphism. Thus L is the lattice of open subspaces of $S(L)$.*

Proof. Let $x_a \in L$, $i_a(z) = z \wedge x_a$, $i(z) = z \wedge Vx_a$. Define uRv by '$i_a(u) = i_a(v)$ for all a'. Then uRv implies

$$i(u) = u \wedge Vx_a = V(u \wedge x_a) = i(v).$$

On the other hand $i(u) = i(v)$ evidently implies $u \wedge x_a = v \wedge x_a$, since

$$u \wedge x_a = (u \wedge Vx_a) \wedge x_a.$$

It follows that $(C(Vx_a), i)$ is the join of $(C(x_a), i_a)$. Now let $x, y \in L$, $i(z) = z \wedge x$, $j(z) = z \wedge y$ and $k(z) = z \wedge x \wedge y$. It is evident that $(C(x \wedge y), k) < (C(x), i), (C(y), j)$. Suppose that $(k, h) < (C(x), i)$, $(C(y), j)$. Now

$$\begin{aligned} k(u) = k(v) &\Rightarrow u \wedge x \wedge y = V \wedge x \wedge y \\ &\Rightarrow j(u \wedge x) = j(v \wedge x) \\ &\Rightarrow h(u \wedge x) = h(v \wedge x) \\ &\Rightarrow h(u) \wedge h(x) = h(v) \wedge h(x) \\ &\Rightarrow h(u) = h(v) \quad \text{since} \quad h(x) = I \text{ in } K. \end{aligned}$$

Thus $(K, h) < (C(x \wedge y), k)$.

PROPOSITION 5. *$(C(x), i)$ and $(L/C(x), h)$ (h canonical) are unique complements in $S(L)$.*

Proof. (i) Let $(K,f) < (C(x), i) \wedge (L/C(x), h)$. For $u, v \in L$, $f(u) = f(v)$ if either $i(u) = i(v)$ or $h(u) = h(v)$. But $i(x) = I = i(I)$ and $h(x) = 0$. Thus in K, $0 = I$, i.e. $(K,f) = 0$.

(ii) Let $(K,f) = (C(x), i) \vee (L/C(x), h)$. For $u, v \in L$,

$$\begin{aligned} f(u) = f(v) &\Rightarrow [i(u) = i(v) \text{ and } h(u) = h(v)] \\ &\Rightarrow [u \wedge x = v \wedge x \text{ and } u \vee x = v \vee x] \\ &\Rightarrow u = v, \quad \text{since } L \text{ is distributive.} \end{aligned}$$

Thus $K = L$ and f is an automorphism, i.e. $(K,f) = I$, by the rule of identity in $S(L)$.

(iii) Suppose that (K, h') is a complement of $(C(x), i)$. It is easy to see that $h'(x) = 0$. It follows that if $u \vee x = v \vee x$, $h'(u) = h'(v)$. Suppose on the other hand, that $h'(u) = h'(v)$. Then $h'(u \vee x) = h'(v \vee x)$ and, by definition

$$i(u \vee x) = i(v \vee x).$$

Thus $u \vee x = v \vee x$, since the congruence relation corresponding to $(K, h) \vee (C(x), i)$ is, by hypothesis, the equality relation. Hence $h'(u) = h'(v)$ if and only if $u \vee x = v \vee x$ and $(K, h') = (L/C(x), h')$.

(iv) In the same way one can prove that $(L/C(x), h)$ has no other complement.

PROPOSITION 6. (*a*) *The correspondence $(C(x), i) \to (L/C(x), h)$ is a V-anti-isomorphism.*

(*b*) *The joint of any finite number of closed subspaces and any intersection of closed subspaces are closed subspaces.*

Proof. We have to prove

(*a*) $(L/C(x), h) \vee (L/C(y), f) = (L/C(x \wedge y), g)$.

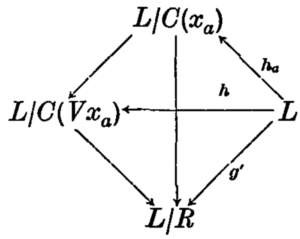
(*b*) $\Lambda(L/C(x_a), h_a) = (L/C(Vx_a), h)$, where h, f, g, h_a are canonical. Let

$$(L/R, g') = (L/C(x), h) \vee (L/C(y), f).$$

Then

$$uRv \Leftrightarrow u \vee x = v \vee x \quad \text{and} \quad u \vee y = v \vee y \Leftrightarrow u \vee (x \wedge y) = v \vee (x \wedge y),$$

so that L/R is $L/C(x \wedge y)$, which proves (a).



Now let $(L/R, g') = \Lambda(L/C(x_a), h_a)$. For each a,

$$u \vee x_a = v \vee x_a \Rightarrow u \vee Vx_a = v \vee Vx_a \quad \text{so that} \quad (L/C(Vx_a), h) < (L/R, g').$$

On the other hand $g'(x_a) = 0$, for all a, so that if $u \vee Vx_a = v \vee Vx_a$, $g'(u) = g'(v)$, i.e. $h(u) = h(v) \Rightarrow g'(u) = g'(v)$. It follows that $(L/R, g') < (L/C(Vx_a), h)$, and we have proved (b).

Definition 3. (a) *For* $s \in S(L)$, $\bar{S} = \Lambda(t \in S(L)$, *closed*, $s < t)$.
(b) *If* $s = (K, h)$ *we write* $\bar{S} = (\bar{K}, \bar{h})$.

Proposition 7. (a) $s = \bar{S}$ *if and only if* s *is closed.*
(b) $s < t$ *implies* $\bar{s} < \bar{t}$.
(c) $\bar{0} = 0$, $\bar{I} = I$.
(d) $\bar{\bar{s}} = \bar{s}$.
(e) $\overline{s \vee t} = \bar{s} \vee \bar{t}$.

Proof. Obvious.

The following propositions, which are easy to prove, illustrate paratopological theorems of subspaces.

Proposition 8. *Let* L *be a quasi-compact. Then*
(a) *If* $s = (K, h)$ *is a closed subspace of* L, K *is quasi-compact.*
(b) *If* (S_a) *is a family of subspaces with the finite intersection property in the order of* $S(L)$, $\Lambda S_a \neq 0$.
(c) *A finite join of quasi-compact subspaces is quasi-compact.*

Definition. *A CBL is connected if no element except* 0 *and* I *has a complement. A subspace* $s = (K, h)$ *is connected if* K *is connected.*

Proposition 9. (a) *The closure of a connected subspace is connected.*
(b) *If* $s \wedge t \neq 0$ *in* $S(L)$ *and if* s *and* t *are connected subspaces, then* $s \vee t$ *is connected.*

4. *Products and Tychonoff's theorem.* The notation of product of topological spaces can be carried over in various ways to CBL. Of these the most interesting treats products as category properties, the essential feature of the product $X = \pi X_\alpha$ of a family of spaces X_α being that any set of continuous functions $f_\alpha: Y \to X_\alpha$ uniquely determines a function $f: Y \to X$ such that $f_\alpha = fp_\alpha$, where p_α is the projection $p_\alpha: X \to Y_\alpha$. These remarks suggest the following definitions, bearing in mind that the operator $L(\ )$ is a *contravariant* functor, which maps the category of topological spaces and continuous functions *into* the category of CBL and V-homomorphisms.

DEFINITION (*a*) *An L-category is a subcategory of the category of CBL and V-homomorphisms which we shall suppose to contain with any CBL, all its V-sublattices.*

(*b*) *Let C be an L-category and* (L_α) *a family of objects of C. We say that the object L of C is the C-product of* (L_α) *if*

(i) *there exists a set* (f_α) *of C-maps (projection maps)* $f_\alpha: L_\alpha \to L$;

(ii) *any C-object K and family* $g_\alpha: L_\alpha \to K$ *of C-maps uniquely determines a C-map* $g: L \to K$ *such that* $gf_\alpha = g_\alpha$.

The C-product of a given family (L_α) is different for different L-categories C, all containing (L_α) and, in particular, the product of a family $L(X_\alpha)$ is not the same in the L-category of *all* CBL as in the L-category of CBL of the form $L(X)$. Nevertheless, interesting theorems on products can be proved without fixing a particular C-category. For example:

PROPOSITION 10. *Let C be an L-category containing the two elements lattice K. A C-product of quasi-compact C-objects* (L_α) *is quasi compact.*

To prove this proposition, we first prove two lemmas.

LEMMA 1. *A CBL is quasi-compact if and only if I is not the joint of all the elements of a co-filter (i.e. an ideal which is not the whole lattice).*

Proof. Let L be quasi-compact and suppose, if possible, that J is a co-filter the join of whose elements is I. There must exist a finite subset $x_1, \ldots, x_n$ of elements of J such that $x_1 \vee x_2 \vee \ldots \vee x_n = I$. But this implies that I is in J, i.e. $J = L$, which is impossible. Conversely, suppose that there is a set (x_α) such that $Vx_\alpha = I$ but for which no finite subset has I as its join. The set of finite joins of elements of (x_α) is then the base of a co-filter the join of whose elements is I.

COROLLARY. *An ultra-co-filter, J, on a quasi-compact CBL contains all joins of its elements, for if not such an element could be used to extend J to form a larger co-filter which would not be the whole lattice.*

LEMMA 2. *Let L be the C-product of* (L_α) *and* $f_\alpha: L_\alpha \to L$ *the projection maps. Then every element of L is the join of finite intersections of elements of the form* $f_\alpha(u_\alpha)$.

Proof. The infinite distributivity of CBL implies that the set L' of joins of finite intersections of elements of the form $f_\alpha(u_\alpha)$ is a V-sublattice of L. By convention, L' is a C-object. For clarity, we use the symbol f'_α for g_α considered as a map into L'. Thus $g_\alpha: L_\alpha \to L'$. By definition of product there is a C-map $g: L \to L'$ such that $gf_\alpha = g_\alpha$. Let f be the injection map $f: L' \to L$ and define $h': L \to L$ by $h' = fg$. Then $h'f_\alpha = fgf_\alpha = fg_\alpha = f_\alpha$. But the definition of product assures us that there is a unique map $g': L \to L$ such that $g'f_\alpha = f_\alpha$. Hence h' is the identity map and $L' = L$ and the lemma is proved.

Proof of Proposition 10. Let (f_α) be the projections of L. Choose an ultra-co-filter (maximal proper ideal), J, on L. We have to prove that $VJ \neq I$. For each α put $J_\alpha = (x \in L_\alpha; f_\alpha(x) \in J)$. J_α is a co-filter; let J'_α be an ultra-co-filter refining it. Since L_α is quasi-compact J'_α contains all joins of its elements. If $y \in J'_\alpha$, it is easily proved that there is an element $x \in J'_\alpha$ such that $x \vee y = I$, i.e. J'_α is prime. Hence, if h_α is the canonical map of $L \to L/J'_\alpha$, $h_\alpha(y) = I$, since $h_\alpha(x) = 0$. It follows that $L/J'_\alpha = K$, the two-element lattice, and $h_\alpha(z) = 0$ or I depending on whether $z \in J'_\alpha$ or not. That h_α is a V-homomorphism follows from the fact that J'_α is a V-ideal, by the corollary of Lemma 1. Since L is the product of the L_α, there is an h such that $h_\alpha = hf_\alpha$. We shall show that h carries J into 0, so that $h(VJ) = 0$, which implies $VJ \neq I$. Every element of L is the join of finite intersections of elements of the form $f_\alpha(u)$. But the prime property of J implies that it must be generated by elements $f_\alpha(u)$ which themselves belong to J, i.e. by $f_\alpha(u)$ with $u \in J_\alpha$. But then $h_\alpha(u) = 0$, so that $hf_\alpha(u) = 0$. Thus $h(x) = 0$ for all $x \in J$. This completes the proof.

I wish to acknowledge the generous encouragement of Prof. C. Ehresmann and discussions with J. Benabou and D. Papert. The term paratopology is borrowed from Ehresmann but is used here in a different sense.