The Mathematical Unconscious

Seymour Papert

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It is deeply embedded in our culture that the appreciation of mathematical beauty and the experience of mathematical pleasure are accessible only to a minority, perhaps a very small minority, of the human race. This belief is given the status of a theoretical principle by Henri Poincaré, who has to be respected not only as one of the seminal mathematical thinkers of the century but also as one of the most thoughtful writers on the epistemology of the mathematical sciences. Poincaré differs sharply from prevalent trends in cognitive and educational psychology in his view of what makes a mathematician. For Poincaré the distinguishing feature of the mathematical mind is not logical but aesthetic. He also believes, but this is a separate issue, that this aesthetic sense is innate: some people happen to be born with the faculty of developing an appreciation of mathematical beauty, and those are the ones who can become creative mathematicians. The others cannot.

This essay uses Poincaré's theory of mathematical creativity as an organizing center for reflections on the relationship between the logical and the extra logical in mathematics and on the relationship between the mathematical and the nonmathematical in the spectrum of human activities. The popular and the sophisticated wings of our culture almost unanimously draw these dichotomies in hard-edged lines. Poincaré's position is doubly interesting because in some ways he softens, and in some ways sharpens, these lines. They are softened when he attributes to the aesthetic an important functional role in mathematics. But the act of postulating a specifically mathematical aesthetic, and particularly an innate one, sharpens the separation between the mathematical and the nonmathematical. Is the mathematical aesthetic really different? Does it have common roots with other components of our aesthetic system? Does mathematical pleasure draw on its own pleasure principles or does it derive from those that animate other phases of human life? Does mathematical intuition differ from common sense in nature and form or only in content?

These questions are deep, complex, and ancient. My daring to address them in the space of a short essay is justified only because of certain simplifications. The first of these is a transformation of the questions similar in spirit to Jean Piaget's way of transforming philosophical questions into psychogenetic ones to which experimental investigations into how children think become refreshingly relevant. By so doing, he has frequently enraged or bewildered philosophers, but has enriched beyond measure the scientific study of mind. My transformation turns Poincaré's theory of the highest mathematical creativity into a more mundane but more manageable theory of ordinary mathematical (and possibly nonmathematical) thinking.

Bringing his theory down to earth in this way possibly runs the risk of abandoning what Poincaré himself might have considered to be most important. But it makes the theory more immediately relevant, perhaps even quite urgent, for psychologists, educators, and others. For example, if Poincaré's model turned out to contain elements of a true account of ordinary mathematical thinking, it could follow that mathematical education as practiced today is totally misguided and even self-defeating. If mathematical aesthetics gets any attention in the schools, it is as an epiphenomenon, an icing on the mathematical cake, rather than as the driving force which makes mathematical thinking function. Certainly the widely practiced theories of the psychology of mathematical development (such as Piaget's) totally ignore the aesthetic, or even the intuitive, and concentrate on structural analysis of the logical facet of mathematical thought.

The destructive consequences of contemporary mathematics teaching can also be seen as a minor paradox for Poincaré. The fact that schools, and our culture generally, are so far from being nurturant of nascent mathematical aesthetic sense in children causes Poincaré's major thesis about the importance of aesthetics to undermine his grounds for believing in his minor thesis which asserts the innateness of such sensibilities. If Poincaré is right about aesthetics, it becomes only too easy to see how the apparent rareness of mathematical talent could be explained without appeal to innateness.

These remarks are enough to suggest that the mundane transformation of Poincaré's theory might be a rich prize for educators even if it lost all touch with the processes at work in big mathematics. But perhaps we can have the best of both worlds. By adopting, as we shall, a more experiential mode of discussion through which theories about mathematical thinking can be immediately confronted with the reader's own mental processes, we do not, of course, renounce the possibility that the mathematical elite share similar experiences. On the contrary, that part of Poincaré's thinking which will emerge as most clearly valid in the ordinary context resonates strongly with modern trends which, in my view, constitute a paradigm shift in thinking about the foundations of mathematics. The concluding paragraphs of my essay will illustrate this resonance in the case of the Bourbaki theory of the structure of mathematics.

My goal here is not to advance a thesis with crisp formulations and rigorous argument, and it is certainly not to pass judgment on the correctness of Poincaré's theory. I shall be content (th1s 1s my second major simplification) to suggest to nonmathematical readers perceptions of, and a discourse about, mathematics which will place it closer than is commonly done to other experiences they know and enjoy. The major obstacle to doing so is a projection of mathematics which greatly exaggerates its logical face much as the Mercator projection of the globe exaggerates the polar regions so that on the map northern Greenland becomes more imposing than equatorial Brazil. Thus our discussion w1ll polarize around separating and relating what I shall call the extra logical face of mathematics from the logical face. I shall ignore distinctions which ought to be made within these categories. Mathematical beauty, mathematical pleasure, and even mathematical intuition will be treated almost interchangeably insofar as they are representatives of the extra logical. And, on the other side, we shall not separate such very different facets of the logical as the formalists' emphasis on the deductive process, Russell's reductionist position (against which Poincaré fought so savagely), and Tarski's set theoretic semantics. These logical theories can be thrown together insofar as they have in common an intrinsic autonomous view of mathematics. They deal with mathematics as self-contained, as justifying itself by formally defined (that is, mathematical) criteria of validity, and they ignore all reference of mathematics to anything outside itself. They certainly ignore phenomena of beauty and pleasure.

There is no theoretical tension in the fact that mathematical logicians ignore, as long as they do not deny, the extra logical. No one will call into question either the reality of the logical face of mathematics or the reality of mathematical beauty or pleasure. What Poincaré challenges is the possibility of understanding mathematical activity, the work of the mathematician solely, or even primarily, in logical terms without reference to the aesthetic. Thus his challenge is in the field of psychology, or the theory of mind and as such, has wider reverberations than the seemingly specialized problem of understanding mathematical thinking: his challenge calls in question the separation within psychology of cognitive functions, defined by opposition to considerations of affect, of feeling, of sense of beauty.

1 shall, on the whole, side with Poincaré against the possibility of a "purely cognitive" theory of mathematical thinking but express reservations about the high degree of specificity he attributes to the mathematical. But first I must introduce another of the themes of Poincaré's theory. This is the role and the nature of the unconscious.

As the aesthetic vs. the logical leads us to confront Poincaré with the cognitive psychology, so the unconscious vs. the conscious leads to a confrontation with Freud. Poincaré is close to Freud in clearly postulating two minds (the conscious and the unconscious), each governed by its own dynamic laws, each able to carry out different functions with severely limited access to the other's activities. As we shall see, Poincaré is greatly impressed by the way in which the solution to a problem on which one has been working at an earlier time often comes into consciousness unannounced, and almost ready-made, as if produced by a hidden part of the mind. But Poincaré's unconscious is very different from Freud's. Far from being the site of prelogical, sexually charged, primary processes, it is rather like an emotionally neutral, supremely logical, combinatoric machine.

The confrontation of these images of the unconscious brings us back to our questions about the nature of mathematics itself. The logical view of mathematics is definitionally discorporate, detached from the body and molded only by an internal logic of purity and truth. Such a view would be concordant with Poincaré's neutral unconscious rather than with Freud's highly charged, instinct ridden dynamics. But Poincaré himself, as I have already remarked, rejects this view of mathematics; even if it could be maintained (which is already dubious) as an image of the finished mathematical product, it is totally inadequate as an account of the productive process through which mathematical truths and structures emerge. In its most naive form the logical image of mathematics is a deductive system in which new truths are derived from previously derived truths by means of rigorously reliable rules of inference. Although less naive logicist theses cannot be demolished quite so easily, it is relevant to notice the different ways in which this account of mathematics can be criticized. It is certainly incomplete since it fails to explain the process of choice determining how deductions are made and which of those made are pursued. It is misleading in that the rules of inference actually used by mathematicians would, if applied incautiously, quickly lead to contradictions and paradoxes. Finally, it is factually false as a description in that it provides no place for the as yet undebugged partial results with which the actual mathematician spends the most time. Mathematical work does not proceed along the narrow logical path of truth to truth to truth, but bravely or gropingly follows deviations through the surrounding marshland of propositions which are neither simply and wholly true nor simply and wholly false.

Workers in artificial intelligence have patched up the first of these areas of weakness, for example, by formalizing the process of setting and managing new problems as part of the work of solving a given one. But if the new problems and the rules for generating them are cast in logical terms, we see this as, at best, the replacement of a static logic by a dynamic one. It does not replace logic by something different. The question at issue here is whether even in the course of working on the most purely logical problem the mathematician evokes processes and sets problems which are not themselves purely logical.

The metaphor of wandering off the path of truth into surrounding marshlands has the merit, despite its looseness, of sharply stating a fundamental problem and preoccupation of Poincaré's: the problem of guidance, or, one might say, of "navigation in intellectual space." If we are content to churn out logical consequences, we would at least have the security of a safe process. In reality, according to Poincaré, the mathematician is guided by an aesthetic sense: in doing a job, the mathematician frequently has to work with propositions which are false to various degrees but does not have to consider any that offend a personal sense of mathematical beauty.

Poincaré's theory of how the aesthetic guides mathematical work divides the work into three stages. The first is a stage of deliberate conscious analysis. If the problem is difficult, the first stage will never, according to Poincaré, yield the solution. Its role is to create the elements out of which the solution will be constructed. A stage of unconscious work, which might appear to the mathematician as temporarily abandoning the task or leaving the problem to incubate, has to intervene. Poincaré postulates a mechanism for the incubation. The phenomenological view of abandonment is totally false. On the contrary, the problem has been turned over to a very active unconscious which relentlessly begins to combine the elements supplied to it by the first, conscious stage of the work. The unconscious mind is not assumed to have any remarkable powers except concentration, systematic operation, and imperviousness to boredom, distractions, or changes of goal. The product of the unconscious work is delivered back to the conscious mind at a moment which has no relation to what the latter is doing. This time the phenomenological view is even more misleading since the finished piece of work might appear in consciousness at the most surprising times, in apparent relation to quite fortuitous events.

How does the unconscious mind know what to pass back to the conscious mind? It is here where Poincaré sees the role of the aesthetic. He believes, as a matter of empirical observation, that ideas passed back are not necessarily correct solutions to the original problem. So he concludes that the unconscious is not able to rigorously determine whether an idea is correct. But the ideas passed up do always have the stamp of mathematical beauty. The function of the third stage of the work is to consciously and rigorously examine the results obtained from the unconscious. They might be accepted, modified, or rejected. In the last case the unconscious might once more be called into action. We observe that the model postulates a third agent in addition to the conscious and unconscious minds. This agent is somewhat akin to a Freudian censor; its job is to scan the changing kaleidoscope of unconscious patterns allowing only those which satisfy its aesthetic criteria to pass through the portal between the minds.

Poincaré is describing the highest level of mathematical creativity, and one cannot assume that more elementary mathematical work follows the same dynamic processes. But in our own striving toward a theory of mathematical thinking we should not assume the contrary either, and so it is encouraging to see even very limited structural resemblances between the process as described by Poincaré and patterns displayed by nonmathematicians whom we ask to work on mathematical problems in what has come, at MIT, to be called "Loud Thinking," a collection of techniques designed to elicit productive thought (often in domains, such as mathematics, they would normally avoid) and make as much of it as possible explicit. The example that follows illustrates aspects of what the very simplest kind of aesthetic guidance of thought might be. The subjects in the experiment clearly proceed by a combinatoric such as that which Poincaré postulates in his second stage until a result is obtained which is satisfactory on grounds that have at least as much claim to be called aesthetic as logical. The process does differ from Poincaré's description in that it remains on the conscious level. This could be reconciled with Poincaré's theory in many ways: one might argue that the number of combinatorial actions needed to generate the acceptable result is too small to require passing the problem to the unconscious level, or that these non-mathematicians lack the ability to do such work unconsciously. In any case, the point of the example (indeed, of this essay as a whole) is not to defend Poincaré in detail but to illustrate the concept of aesthetic guidance.

The problem on which the subjects were asked to work was the proof that the square root of 2 is irrational. The choice is particularly appropriate here because this theorem was selected by the English mathematician G. H. Hardy as a prime example of mathematical beauty, and consequently it is interesting, in the context of a nonelitist discussion of mathematical aesthetics, to discover that many people with very little mathematical knowledge are able to discover the proof if emotionally supportive working conditions encourage them to keep going despite mathematical reticence. The following paragraphs describe an episode through which almost all the subjects in our investigation pass. To project ourselves into this episode, let us suppose that we have set up the equation:

$\sqrt{2}$=p/q where p and q are whole numbers

Let us also suppose that we do not really believe that 2 can be so expressed. To prove this, we seek to reveal something bizarre; in fact contradictory, behind the impenetrably innocent surface Impression of the equation. We clearly have to do with an interplay of latent and manifest contents. What steps help in such cases?

Almost as if they had read Freud, many subjects engage in a process of mathematical "free association" trying in turn various transformations associated with equations of this sort. Those who are more sophisticated mathematically need a smaller number of tries, but none of the subjects seem to be guided by a prevision of where the work will go. Here are some examples of transformations in the order they were produced by one subject:

$\sqrt{2} = p/q$

$\sqrt{2} \times q = p$

$P = \sqrt{2} \times q$

$(\sqrt{2})^2 = (p/q)^2$

$2 = p^2/q^2$

$P^2 = 2q^2$

All subjects who have become more than very superficially involved in the problem show unmistakable signs of excitement and pleasure when they hit on the last equation. This pleasure is not dependent on knowing (at least consciously) where the process IS leading. It happens before the subjects are able to say what they will do next, and, in fact, it happens even in cases where no further progress is made at all. And the reaction to $p^2 = 2q^2$ is not merely affective· once this has been seen, the subjects scarcely ever look back at any of the earlier transforms or even at the original equation. Thus there is something very special about $p^2 = 2q^2$ What is it? We first concentrate on the fact that it undoubtedly has a pleasurable charge and speculate about the sources of the charge. What is the role of pleasure mathematics?

Pleasure is, of course, often experienced in mathematical work, as if one were rewarding oneself when one achieves a desired goal after arduous struggle. But it is highly unplausible that this actual equation was anticipated here as a preset goal. If the pleasure was that of goal achievement, the goal was of a very different, less formal 1 would say "more aesthetic" nature than the achievement of a particular equation. To know exactly what it is would require much more knowledge about the individual subjects than we can include here. It is certainly different from subject to subject and even multiply overdetermined in each subject. Some subjects explicitly set themselves the goal: "get rid of the square root." Other subjects did not seem explicitly to set themselves this goal but were nevertheless pleased to see the square root sign go away. Others, again, made no special reaction to the appearance of $2=p^2/q^2$ until this turned into $p^2 = 2q^2$. My suggestion is that the elimination of the root sign for the obvious simple instrumental purpose is only part of a more complex story: the event is resonant with several processes which might or might not be accessible to the conscious mind and might or might not be explicitly formulated as goals. I suggest, too, that some of these processes tap into other sources of pleasure, more specific and perhaps even more primitive than the generalized one of goal attainment. To make these suggestions more concrete, I shall give two examples of such pleasure giving processes.

The first example is best described in terms of the case frame type of calculus of situations characteristic of recent thinking in artificial intelligence. The original equation is formalized as a situation frame with case slots for "three actors," of which the principle or "subject" actor is $\sqrt{2}$. The other two actors, p and q, are subordinate dummy actors whose roles are merely to make assertions about the subject actor. When we turn the situation into $p^2 = 2q^2$, it is as sharply different as in a figure/ground reversal or the replacement of a screen by a face in an infant's perception of peek-a-boo. Now p has become the subject, and the previous subject, $\sqrt{2}$, has vanished. Does this draw on the pleasure sources that make infants universally enjoy peek-a-boo?

The other example of what might be pleasing in this process comes from the observation that 2 has not vanished away completely without trace. The 2 is still visible in $p^2 = 2q^2$. However, these two occurrences of 2 are so very different in role that identifying them gives the situation a quality of punning, or "condensation" at least somewhat like that which Freud sees as fundamental to the effectiveness of wit. The attractiveness and plausibility of this suggestion comes from the possibility of seeing condensation in very many mathematical situations. Indeed, the very central idea of abstract mathematics could be seen as condensation: the "abstract" description simultaneously signifies very different "concrete" things. Does this allow us to conjecture that mathematics shares more with jokes, dreams, and hysteria than is commonly recognized?

It is of course dangerous to go too far in the direction of presenting the merits of $p^2 = 2q^2$ in isolation from its role in achieving the original purpose, which was not to titillate the mathematical pleasure senses but to prove that 2 is irrational. The statement of the previous two paragraphs needs to be melded with an understanding of how the work comes to focus on $p^2 = 2q^2$ through a process not totally independent of recognizing it as a subgoal of the supergoal of proving the theorem.

How do we integrate the functional with the aesthetic? The simplest gesture in this direction for those who see the eminently functional subgoal system as the prime mover is to enlarge the universe of discourse in which subgoals can be formulated. Promoting a subordinate character (that is, p) on the problem scene to a principal role is, within an appropriate system of situation frames, as well-defined a subgoal as, say, finding the numerical solution of an equation. But we are now talking about goals which have lost their mathematical specificity and may be shared with nonmathematical situations of life or literature. Taken to its extreme, this line of thinking leads us to see mathematics, even in its detail, as an acting out of something else: the actors may be mathematical objects, but the plot is spelled out in other terms. Even in it's less extreme forms this shows how the aesthetic and the functional can enter into a symbiotic relationship of, so to speak, mutual exploitation. The mathematically functional goal is achieved through a play of subgoals formulated in another, nonmathematical discourse, drawing on corresponding extra mathematical knowledge. Thus the functional exploits the aesthetic. But to the extent we see (here in a very Freudian spirit) the mathematical process itself as acting out premathematical processes, the reverse is also true.

These speculations go some (very little) way toward showing how Poincaré's mathematical aesthetic sentinel could be reconciled with existing models of thinking to the enrichment of both. But the attempt to do so very sharply poses one fundamental question about the relationship between the functional and the aesthetic and hedonistic facets not only of mathematics but of all intellectual work. What is it about each of these that makes it able to serve the other? Is it not very strange that knowledge, or principles of appreciation, which is useful outside of mathematics, should have application within? The answer must lie in a genetic theory of mathematics. If we adopt a Platonic (or logical) view of mathematics as existing independently of any properties of the human mind, or of human activity, we are forced to see such interpretations as highly unlikely. In my remaining pages I shall touch on a few more examples of how mathematics can be seen from a perspective which makes its relationship to other human structures more natural. We begin by looking at another episode of the story about the square root of 2.

Our discussion of $p^2 = 2q^2$ was almost brutally nonteleological in that we discussed it from only one side, the side from which it came, pretending ignorance of where it is going. We now remedy this by seeing how it serves the original intention of the work which was to find a contradiction in the assumption $2= p/q$. It happens that there are several paths one can take to this goal. Of these 1 shall contrast two which differ along a dimension one might call "gestalt vs. atomistic" or "aha-single-flash-insight vs. step-by-step reasoning." The step-by-step form is the more classical (it is attributed to Euclid himself) and proceeds in the following manner. We can read off from $p = 2q$ that p is even. It follows that p is even. By definition this means that is twice some other whole number which we can call r. So

$p = 2r$

$p^2 = 4r^2$

$4r^2 = 2q^2$ (remember: $p^2 = 2q^2$)

$q^2 = 2r^2$

and we deduce that q is also even. But this at last really is manifestly bizarre since we chose p and q in the first place and could, had we wished, have made sure that they had no common factor.

So there is a contradiction. Before commenting on the aesthetics of this process, we look at the "flash" version of the proof. It depends on having a certain perception of whole numbers, namely, as unique collections of prime factors: 6 = 3 × 2 and 36 = 3 × 3 × 2 × 2. If you solidly possess this frame for perceiving numbers, you probably have a sense of immediate perception of a perfect square (36 or p^2 or q^2) as an even set. If you do not possess it, we might have to use step-by-step arguments (such as let $p = p_1p_2 \ldots p_k$, so that $p^2 = p_1p_1p_2p_2 \ldots p_kp_k$), and this proof then becomes, for you-here-and-now, even more atomistic and certainly less pleasing than the classical form. But if you do see (or train yourself to see) p and q as even sets, you will also see $p= 2q$ as making the absurd assertion that an even set (p^2) is equal to an odd set (q^2 and one additional factor: 2). Thus, given the right frames for perceiving numbers, $p^2 = 2q^2$ is (or so it appears phenomenologically) directly perceived as absurd.

Although there is much to say about the comparative aesthetics of these two little proofs, I shall concentrate on just one facet of beauty and pleasure found by some subjects in our experiments. Many people are impressed by the brilliance of the second proof. But if this latter attracts by its cleverness and immediacy, it does not at all follow that the first loses by being (as I see it) essentially serial. On the contrary, there is something very powerful in the way one is captured and carried inexorably through the serial process. I do not merely mean that the proof is rhetorically compelling when presented well by another person, although this is an important factor in the spectator sport aspect of mathematics. I mean rather that you need very little mathematical knowledge for the steps to be forced moves, so that once you start on the track you will find that you generate the whole proof.

One can experience the process of inevitability in very different ways with very different kinds of affect. One can experience it as being taken over in a relationship of temporary submission. One can experience this as surrender to Mathematics, or to another person, or of one part of oneself to another. One can experience it not as submission but as the exercise of an exhilarating power. Any of these can be experienced as beautiful, as ugly, as pleasurable, as repulsive, or as frightening.

These remarks, although they remain at the surface of the phenomenon, suffice to cast serious doubt on Poincaré's reasons for believing that the faculty for mathematical aesthetic is inborn and independent of other components of the mind. They suggest too many ways in which factors of a kind Poincaré does not consider might, in principle, powerfully influence whether an individual finds mathematics beautiful or ugly and which kinds of mathematics he will particularly relish or revile. To see these factors a little more clearly, let us leave mathematics briefly to look at an example from a very sensitive work of fiction: Robert Pirsig's Zen and the Art of Motorcycle Maintenance. The book is a philosophical novel about different styles of thought. The principal character, who narrates the events, and his friend John Sutherland are on a motorcycling vacation which begins by riding from the east coast to Montana. Sometime before the trip recounted in the book, John Sutherland had mentioned that his handlebars were slipping. The narrator soon decided that some shimming was necessary and proposed cutting shim stock from an aluminium beer can. "I thought this was pretty clever myself," he says, describing his surprise at Sutherland's reaction which brought the friendship close to rupture. To Sutherland the idea was far from clever; it was unspeakably offensive. The narrator explains: "I had had the nerve to propose repair of his new eighteen-hundred-dollar BMW, the pride of a half-century of German mechanical finesse, with a piece of old beer can!" But for the narrator there is no conflict; on the contrary: "beer can aluminum is soft and sticky as metals go. Perfect for the application ... in other words any true German mechanic with half a century of mechanical finesse behind him, would have concluded that this particular solution to this particular technical problem was perfect." The difference proves to be unbridgeable and emotionally explosive. The friendship is saved only by a tacit agreement never again to discuss maintenance and repair of the motorcycles even though the two friends are close enough to one another and to their motorcycles to embark together on the long trip described in the book.

Sutherland's reaction would be without consequence for our problem if it showed stupidity, ignorance: or an idiosyncratic quirk about ad hoc solutions to repair problems. But it goes deeper than any of these. Pirsig's accomplishment is to show us the coherence in many such incidents. This accomplishment is quite impressive. Pirsig presents us with materials so rich that we can use them to appreciate kinds of coherence implicit in them which are rather different from the one advanced by Pirsig himself. Here I want to touch briefly on two analogies between the story of Sutherland and the shim stock and issues we have discussed about mathematics: first, the relationship between aesthetics and logic in thinking about mathematics as well as motorcycles, and second, the lines of continuity and discontinuity between mathematics or motorcycles and everything else.

It is clear from the shim stock incident itself, and much more so from the rest of the book, that the continuity for Pirsig's characters between man, machine, and natural environment are very different and that these differences deeply affect their aesthetic appreciation. For the narrator, the motorcycle is continuous with the world not only of beer cans but more generally the world of metals (taken as substance). In this world, the metal's identity is not reducible to a particular embodiment of the metal in a motorcycle or in a beer can. Nor can any identity be reduced to a particular instance of it. For Sutherland, on the contrary, this continuity is not merely invisible, but he has a strong investment in maintaining the boundaries between what the narrator sees as superficial manifestations of the same substance.

For Sutherland, the motorcycle is not only in a world apart from beer cans; it is even in a world apart from other machines, a fact that enables him to relate without conflict to this piece of technology as a means to escape from technology. We could deepen the analysis of the investments of these two characters in their respective positions by noting their very different involvements in work and society. The narrator is part of industrial society (he works for a computer company) and is forced to seek his own identity (as he seeks the identity of metal) in a sense of his substance which lies beyond the particular form into which he has been molded. Like malleable metal, he is something beyond and perhaps better than the form which is now imposed on him. He certainly does not define himself as a writer of computer manuals. His friend Sutherland on the other hand is a musician and is much more able to take his work as that which structures his image of himself in the same way that he takes a motorcycle as a motorcycle and a beer can as a beer can.

We need not pursue these questions of essence and accident much further to make the important point, and a point which is widely ignored: if styles of involvement with motorcycle maintenance are intricated with such complexity with our psychological and social identities, one would scarcely expect this to be less true about the varieties of involvements of individuals with mathematics.

These ideas about the relationship of mathematical work with the whole person can be further illuminated by an example of an experiment in education, Turtle Geometry, as it is used with the LOGO programming language. These experiments express a critique of traditional school mathematics (which applies no less to the so-called new math than to the old). A description of this traditional mathematics in terms of the concepts we have developed in this essay would reveal it to be a caricature of mathematics in its depersonalized, purely logical, “formal” incarnation. Although we can document progress in the rhetoric of math teachers (teachers of the new math are taught to speak in terms of “understanding” and “discovery”), the problem remains because of what they are teaching.

In Turtle Geometry we create an environment in which the child’s task is not to learn a set of formal rules but to develop sufficient insight into the way he moves in space to allow the transposition of this self-knowledge into programs that will cause a cybernetic animal, the turtle, to move. What is a Turtle? It can take several forms. It can be a machine controlled by a computer that crawls on the floor with a pen that leaves a trace of where it has been, or it can be a “light Turtle” which makes tracings on a computer display screen. The literature and ongoing research about the enterprise of developing Turtle Geometry and LOGO environments is more than we can summarize here, but what we want to do is underscore two closely related aspects of Turtle Geometry which are directly relevant to the concerns of this paper. The first is the development of an ego syntonic mathematics, indeed, of a body syntonic mathematics; the second is the development of a context for mathematical work where the aesthetic dimension (even in its narrowest of “the pretty”) is continually placed in the forefront.

We shall give a single example which illuminates both of these aspects; an example of a typical problem that arises when a child is learning Turtle Geometry. The child has already learned how to command the Turtle to move forward in the direction that it is facing and to pivot around its axis, that is, to turn the number of degrees right or left that the child has commanded. With these commands the child has written programs which cause the Turtle to draw straight line figures. Sooner or later the child poses the question: “How can I make the Turtle draw a circle?’’ In Turtle Geometry we do not provide” answers.” Learners are encouraged to use their own bodies to find a solution. The child begins to walk in circles and discovers how to make a circle by going forward a little and turning a little, by going forward a little and turning a little. Now the child knows how to make the Turtle draw a circle: simply give the Turtle the same commands one would give oneself. Expressing “go forward a little, turn a little” comes out in Turtle Language as REPEAT [FORWARD 1 RIGHT-TURN 1]. Thus we see a process of geometrical reasoning that is both ego syntonic and body syntonic. And once the child knows how to place circles on the screen with the speed of light, an unlimited palette of shapes, forms, and motion has been opened. Thus the discovery of the circle (and, of course, the curve) is a turning point in the child’s ability to achieve a direct aesthetic experience through mathematics.

In the above paragraph it sounds as though ego syntonic mathematics was recently invented at MIT. This is certainly not the case and, indeed, would contradict the point made repeatedly in this essay that the mathematics of the mathematician is profoundly personal. It is also not the case that we have invented ego syntonic mathematics for children. We have merely given children a way to re-appropriate what was theirs to begin. Most people feel that they have no "personal" involvement with mathematics, yet as children they constructed it for themselves. Jean Piaget's work on genetic epistemology teaches us that from the first days of life a child is engaged in an enterprise of extracting mathematical knowledge from the intersection of body with environment. The point is that, whether we intend it or not, the teaching of mathematics, as it is traditionally done in our schools, is a process by which we ask the child to forget the natural experience of mathematics in order to learn a new set of rules.

This same process of forgetting extralogical roots has until very recently dominated the formal history of mathematics in the academy. In the early part of the twentieth century, formal logic was seen as synonymous with the foundation of mathematics. Not until Bourbaki's structuralist theory appeared do we see an internal development in mathematics which opens mathematics up to "remembering" its genetic roots. This "remembering" was to put mathematics in the closest possible relationship to the development of research about how children construct their reality.

The consequences of these currents and those we encountered earlier from cognitive and dynamic psychology place the enterprise of understanding mathematics at the threshold of a new period heralded by Warren McCulloch's epigrammatic assertion that neither man nor mathematics can be fully grasped separately from the other. When asked what question would guide his scientific life, McCulloch answered: "What is a man so made that he can understand number and what is number so made that a man can understand it?"

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